Globalni ekstremi

Iščemo največjo (maksimum) oz. najmanjšo (minimum) vrednost funkcije na zaprtem intervalu.

Kandidati:
- stacionarne točke
- krajišča intervala

1. Poišči največjo in najmanjšo vrednost, ki jo zavzame funkcija $f(x) = 3x^5 - 5x^3$ na intervalu $[-\frac{4}{3}, 2]$.

stacionarne točke: $f'(x) = 0$

$$f'(x) = 15x^4 - 15x^2 = 0$$
$$15x^2(x^2-1) = 0$$
$$15x^2(x-1)(x+1) = 0$$
$$x_{1,2} = 0 \quad x_3 = 1 \quad x_4 = -1$$

x	$f(x)$
$-\frac{4}{3}$	$f(-\frac{4}{3}) = 3(-\frac{4}{3})^5 - 5(-\frac{4}{3})^3 \doteq -0.7$
2	$f(2) = 56$
0	0
1	-2
-1	2

MAX $T(2, 56)$

MIN $T(1, -2)$

Integral

Nedoločeni integral funkcije f:

$$F(x) = \int f(x)\,dx$$, za katero velja
$$F'(x) = f(x).$$

Pravili za integriranje:

$$\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int c\cdot f(x)\,dx = c\int f(x)\,dx$$

Elementarni integrali:

- $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$
- $\int \cos x\,dx = \sin x + C$
- $\int \sin x\,dx = -\cos x + C$
- $\int e^x\,dx = e^x + C$
- $\int \frac{1}{\cos^2 x}\,dx = \tan x + C$
- $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$
- $\int \frac{dx}{1+x^2} = \operatorname{arc}\tan x + C$
- $\int \frac{1}{x}\,dx = \log|x| + C$
- $\int 1\,dx = \int dx = x + C$
- $\int \frac{1}{x\pm a}\,dx = \log|x\pm a| + C$

2. Izračunaj naslednje nedoločene integrale:

(a) $\int(3x^2 - 5x - \frac{1}{\sqrt{1-x^2}} + 1 - \cos x)dx$

(b) $\int(\sin x + \frac{2}{x^2} - \frac{1}{x})dx$

(c) $\int(x^6 - 2)^2\,dx$

(d) $\int(\frac{1}{\cos^2 x} - \frac{1}{1+x^2} + 5e^x)dx$

(e) $\int \sin(3x)dx$

(f) $\int \frac{dx}{5x-2}$

(g) $\int \frac{dx}{e^{2x}}$

(h) $\int \sin^4 x \cos x\,dx$

(i) $\int \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$

(j) $\int \frac{dx}{x\log^2(x)}$

(k) $\int(x^2 - 1)^9 x\,dx$

(l) $\int \tan x\,dx$

(m) $\int \frac{e^x}{e^x-1}\,dx$

(n) $\int xe^{-(x^2+1)}dx$

(o) $\int \frac{x}{\cos^2(x^2)}\,dx$

(p) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

a) $\int(3x^2 - 5x - \frac{1}{\sqrt{1-x^2}} + 1 - \cos x)\,dx =$

$3\cdot\frac{x^3}{3} - 5\cdot\frac{x^2}{2} - \arcsin x + x - \sin x + C$

c) $\int(x^6-2)^2 dx = \int(x^{12} - 4x^6 + 4)\,dx =$

$= \frac{x^{13}}{13} - 4\cdot\frac{x^7}{7} + 4\cdot x + C$

e) $\int \sin(\underbrace{3x}_{t})\,\underbrace{dx} = \int \sin(t)\,\frac{dt}{3} = \frac{1}{3}\int \sin(t)\,dt$

$t = 3x \,/\,'$

$dt = 3\,\underbrace{dx} \quad /:3$

$\frac{dt}{3} = \underbrace{dx}$

$= \frac{1}{3}(-\cos t) + C$

$= -\frac{1}{3}\cos(3x) + C$

f) $\int \frac{dx}{5x-2} = \int \frac{\frac{dt}{5}}{t} = \frac{1}{5}\int \frac{dt}{t} =$

$t = 5x-2 \,/'$
$dt = 5dx \,/:5$
$\frac{dt}{5} = dx$

$= \frac{1}{5}\log|t| + C$

$= \frac{1}{5}\log|5x-2| + C$

g) $\int \frac{dx}{e^{2x}} = \int e^{-2x}dx = \int e^t \frac{dt}{-2} = -\frac{1}{2}\int e^t dt$

$t = -2x \,/'$
$dt = -2\underbrace{dx} \,/:(-2)$
$\frac{dt}{-2} = dx$

$= -\frac{1}{2}e^t + C =$
$= -\frac{1}{2}e^{-2x} + C$

h) $\int \sin^4 x \underbrace{\cos x\, dx} = \int t^4 dt = \frac{t^5}{5} + C$

$t = \sin x \,/'$
$\underbrace{dt} = \underbrace{\cos x\, dx}$

$= \frac{\sin^5 x}{5} + C$

i) $\int \frac{\arcsin x}{\sqrt{1-x^2}}dx = \int t\, dt = \frac{t^2}{2} + C = \frac{\arcsin^2 x}{2} + C$

$t = \arcsin x \,/'$
$dt = \frac{1}{\sqrt{1-x^2}}dx$

a) $\int \frac{x}{\cos^2(x^2)}dx = \int \frac{\frac{dt}{2}}{\cos^2(t)} = \frac{1}{2}\int \frac{dt}{\cos^2(t)} =$

$t = x^2 \,/\, '$
$dt = 2x\,dx \quad /:2$
$\frac{dt}{2} = x\,dx$

$= \frac{1}{2}\tan(t) + C =$

$= \frac{1}{2}\tan(x^2) + C$

PER PARTES

$$\int u\, dv = u \cdot v - \int v\, du$$

odvajamo integriramo

3. Izračunaj naslednje nedoločene integrale z uporabo metode *per partes*:

(a) $\int x \log x \, dx$
(b) $\int (2x-1) \sin x \, dx$
(c) $\int \arctan(x) \, dx$
(d) $\int \arcsin(2x) \, dx$

a) $\int x \log x \, dx = \log x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx =$

$$\left[\begin{array}{ccc} u = \log x & \xrightarrow{\;'\;} & du = \frac{1}{x}dx \\ dv = x\,dx & \xrightarrow{\;\int\;} & v = \frac{x^2}{2} \end{array}\right]$$

$= \log x \cdot \frac{x^2}{2} - \frac{1}{2}\int x\,dx =$

$= \log x \cdot \frac{x^2}{2} - \frac{1}{2} \cdot \frac{x^2}{2} + C$

$= \frac{x^2}{2}\left(\log x - \frac{1}{2}\right) + C$

b) $\int (2x-1)\sin x\,dx = (2x-1)\cdot(-\cos x) - \int -\cos x \cdot 2dx$

$$\left[\begin{array}{lcl} u = 2x-1 & \xrightarrow{\ '\ } & du = 2dx \\ dv = \sin x\,dx & \xrightarrow{\ \int\ } & v = -\cos x \end{array}\right]$$

$$= -(2x-1)\cos x + 2\int \cos x\,dx$$

$$= -(2x-1)\cos x + 2\sin x + C$$

c) $\int \arctan(x)dx = \arctan(x)\cdot x - \int x \cdot \frac{1}{1+x^2} dx$

$$\left[\begin{array}{lcl} u = \arctan(x) & \xrightarrow{\ '\ } & du = \frac{1}{1+x^2}dx \\ dv = dx & \xrightarrow{\ \int\ } & v = x \end{array}\right]$$

$$t = 1+x^2 \,\big|\,'$$
$$dt = 2x\,dx \quad |:2$$
$$\frac{dt}{2} = x\,dx$$

$$= \arctan(x)\cdot x - \int \frac{\frac{dt}{2}}{t} =$$

$$= \arctan(x)\cdot x - \frac{1}{2}\int \frac{dt}{t} =$$

$$= \arctan(x)\cdot x - \frac{1}{2}\log|t| + C =$$

$$= \arctan(x)\cdot x - \frac{1}{2}\log|1+x^2| + C$$

4. Izračunaj nedoločene integrale naslednjih racionalnih funkcij.

(a) $\int \frac{x+6}{(x-1)(x-8)}\,dx$

(b) $\int \frac{x^2}{x+1}\,dx$

(c) $\int \frac{x+3}{x-3}\,dx$

(d) $\int \frac{x^2-1}{x^2+1}\,dx$

a) $\int \frac{x+6}{(x-1)(x-8)}\,dx$

st. števca < st. imenovalca

↓

razcep na parcialne ulomke

$$\frac{x+6}{(x-1)(x-8)} = \frac{A}{x-1} + \frac{B}{x-8} = \frac{A(x-8)+B(x-1)}{(x-1)(x-8)}$$

$$x+6 = A(x-8)+B(x-1)$$

$$\underline{x}+\underline{6} = \underline{A}x - \underline{8A} + \underline{B}x - \underline{B}$$

$$\begin{aligned} 1 &= A+B \\ 6 &= -8A-B \quad \oplus \\ \hline 7 &= -7A \quad /:(-7) \\ A &= -1 \end{aligned}$$

$$1 = -1 + B$$

$$B = 2$$

$$\int \frac{x+6}{(x-1)(x-8)}\,dx = \int \left(\frac{-1}{x-1} + \frac{2}{x-8} \right) dx =$$

$$= -\int \frac{1}{x-1}\,dx + 2\int \frac{1}{x-8}\,dx$$

$$= -\log|x-1| + 2\log|x-8| + C$$

b) $\int \frac{x^2}{x+1}\,dx$

st. števca > st. imenovalca

$\downarrow$

delimo polinoma

$$\begin{array}{l} x^2 : (x+1) = x-1 \\ \underline{-x^2 \mp x} \\ \quad -x \\ \underline{\pm x \pm 1} \\ \quad 1 \end{array}$$

$$\frac{x^2}{x+1} = x-1 + \frac{1}{x+1}$$

$$\int \frac{x^2}{x+1}\,dx = \int \left(x-1+\frac{1}{x+1}\right)dx =$$

$$= \frac{x^2}{2} - x + \log|x+1| + C$$