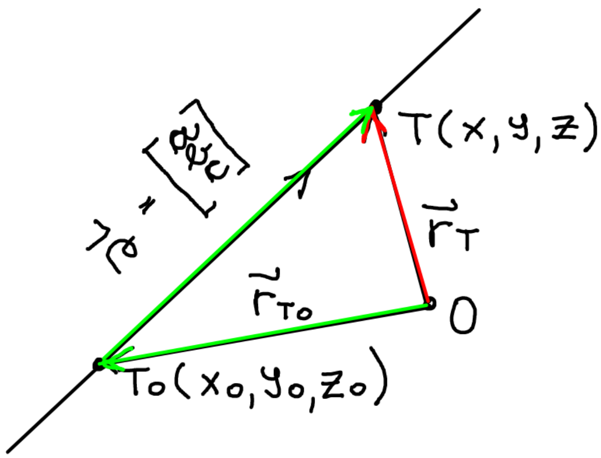
PREMICE IN RAVNINE

Premice



$$\boxed{\vec{r}_T = \vec{r}_{T_0} + t \cdot \vec{a}}$$

$t \in \mathbb{R}$

vektorska oblika enačbe premice

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\boxed{\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \qquad t \in \mathbb{R} \\ z &= z_0 + tc \end{aligned}}$$

parametrična oblika

$$(t =) \boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

kanonična oblika

1. Dane so točke $A(3,2,0)$, $B(2,1,2)$ in $C(4,1,6)$.

   (a) Določi premico p skozi točki A in B. Njeno enačbo zapiši v vseh treh oblikah.

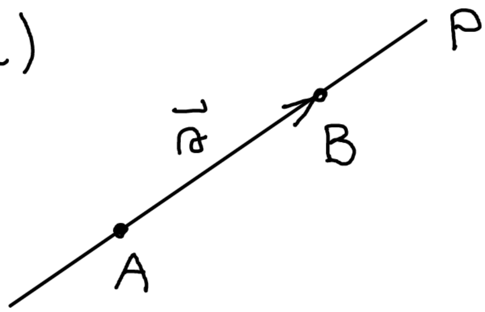
   (b) Ali so točke A, B in C kolinearne?

   (c) Poišči točko D na premici p, tako da bo vektor $\overrightarrow{CD}$ pravokoten na p. Nato določi razdaljo med točko C in premico p.

   (d) Poišči zrcalno sliko C' pri zrcaljenju točke C čez premico p.

   (e) Poišči točki P, Q na premici p, tako da bo $CPC'Q$ kvadrat.

a)



$$\vec{r} = \vec{r}_A + t \cdot \vec{a}, \quad t \in \mathbb{R}$$

$$\vec{a} = \overrightarrow{AB} = \vec{r}_B - \vec{r}_A = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \boxed{\vec{r} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \quad t \in \mathbb{R}}$$

vektorska oblika

$$\boxed{\begin{aligned} x &= 3 - t \\ y &= 2 - t \qquad t \in \mathbb{R} \\ z &= 2t \end{aligned}}$$

parametrična oblika

$$t = \boxed{\frac{x-3}{-1} = \frac{y-2}{-1} = \frac{z}{2}}$$

kanonična oblika

b) A, B, C so kolinearne, če ležijo na isti premici

$C(\underset{x}{4}, \underset{y}{1}, \underset{z}{6}) \to$ vstavimo v kanonično obliko premice skozi A in B

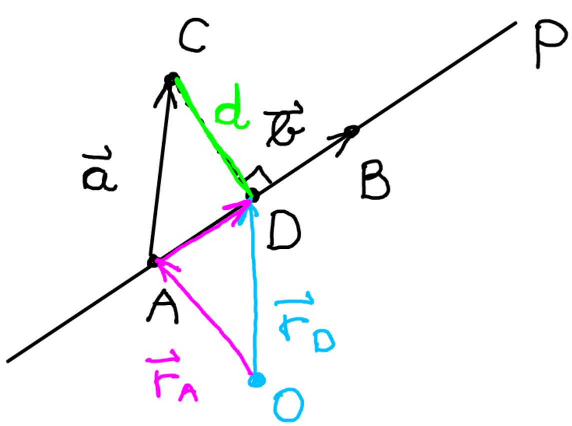
$$\underset{\displaystyle -1}{\underset{\parallel}{\frac{4-3}{-1}}} = \underset{\displaystyle 1}{\underset{\parallel}{\frac{1-2}{-1}}} = \underset{\displaystyle 3}{\underset{\parallel}{\frac{6}{2}}} \quad /\!/ \to$$

točka C ne leži na premici skozi A in B

$\Downarrow$

točke A, B in C niso kolinearne

(c) Poišči točko D na premici p, tako da bo vektor $\overrightarrow{CD}$ pravokoten na p. Nato določi razdaljo med točko C in premico p.



$$\vec{r}_D = \vec{r}_A + \overrightarrow{AD}$$
$$= \vec{r}_A + \mathrm{proj}_{\vec{b}}\, \vec{a}$$

$$\vec{a} = \overrightarrow{AC} = \vec{r}_C - \vec{r}_A = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix}$$

$$\vec{b} = \overrightarrow{AB} = \vec{s} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$$\mathrm{proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\cdot\vec{b} = \frac{1(-1)+(-1)(-1)+6\cdot 2}{\left(\sqrt{(-1)^2+(-1)^2+2^2}\right)^2}\cdot\begin{bmatrix}-1\\-1\\2\end{bmatrix}$$

$$= \frac{12}{6}\begin{bmatrix}-1\\-1\\2\end{bmatrix} = 2\begin{bmatrix}-1\\-1\\2\end{bmatrix} = \begin{bmatrix}-2\\-2\\4\end{bmatrix}$$

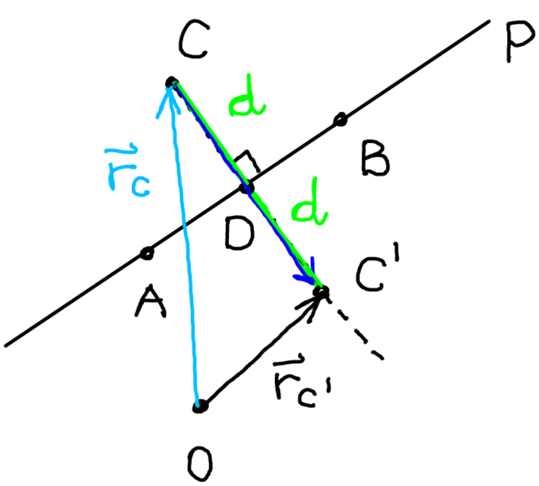
$$\vec{r}_D = \vec{r}_A + \mathrm{proj}_{\vec{b}}\vec{a} = \begin{bmatrix}3\\2\\0\end{bmatrix} + \begin{bmatrix}-2\\-2\\4\end{bmatrix} = \begin{bmatrix}1\\0\\4\end{bmatrix}$$

$$D(1,0,4)$$

$$d = |\overrightarrow{CD}| = \sqrt{(-3)^2+(-1)^2+(-2)^2} = \underline{\underline{\sqrt{14}}}$$

$$\overrightarrow{CD} = \vec{r}_D - \vec{r}_C = \begin{bmatrix}1\\0\\4\end{bmatrix} - \begin{bmatrix}4\\1\\6\end{bmatrix} = \begin{bmatrix}-3\\-1\\-2\end{bmatrix}$$

(d) Poišči zrcalno sliko C' pri zrcaljenju točke C čez premico p.



$$\vec{r}_{C'} = \vec{r}_C + 2\cdot\overrightarrow{CD} =$$

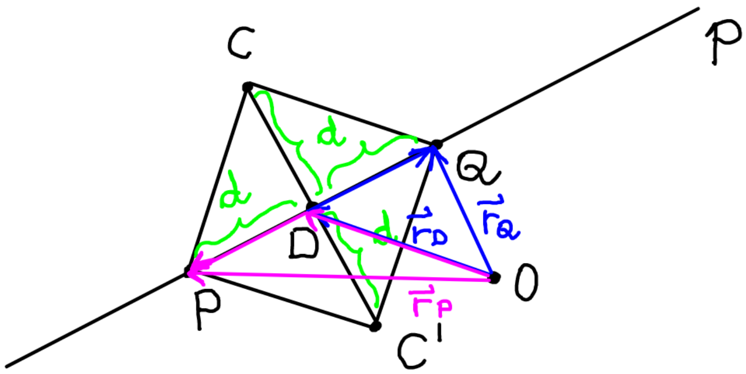
$$= \begin{bmatrix}4\\1\\6\end{bmatrix} + 2\cdot\begin{bmatrix}-3\\-1\\-2\end{bmatrix} =$$

$$= \begin{bmatrix}4\\1\\6\end{bmatrix} + \begin{bmatrix}-6\\-2\\-4\end{bmatrix} =$$

$$= \begin{bmatrix}-2\\-1\\2\end{bmatrix}$$

$$C'(-2,-1,2)$$

(e) Poišči točki P, Q na premici p, tako da bo $CPC'Q$ kvadrat.



$$\vec{r}_Q = \vec{r}_D + \overrightarrow{DQ}$$

$$= \vec{r}_D + \sqrt{14}\,\frac{\vec{a}}{|\vec{a}|}$$

vektor v smeri $\vec{a}$ dolžine 1

$$= \begin{bmatrix}1\\0\\4\end{bmatrix} + \sqrt{14}\,\frac{\begin{bmatrix}-1\\-1\\2\end{bmatrix}}{\sqrt{6}} =$$

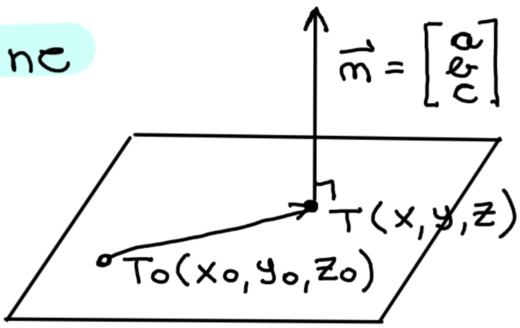
$$= \begin{bmatrix}1\\0\\4\end{bmatrix} + \sqrt{\frac{14}{6}}\begin{bmatrix}-1\\-1\\2\end{bmatrix} =$$

$$= \begin{bmatrix}1\\0\\4\end{bmatrix} + \sqrt{\frac{7}{3}}\begin{bmatrix}-1\\-1\\2\end{bmatrix}$$

$$\vec{r}_P = \vec{r}_D - \sqrt{14}\,\frac{\vec{a}}{|\vec{a}|}$$

$$= \begin{bmatrix}1\\0\\4\end{bmatrix} - \sqrt{\frac{7}{3}}\begin{bmatrix}-1\\-1\\2\end{bmatrix}$$

Ravnine



$$\vec{m} \perp \overrightarrow{T_0T}$$

$$\vec{m} \cdot \overrightarrow{T_0T} = 0$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

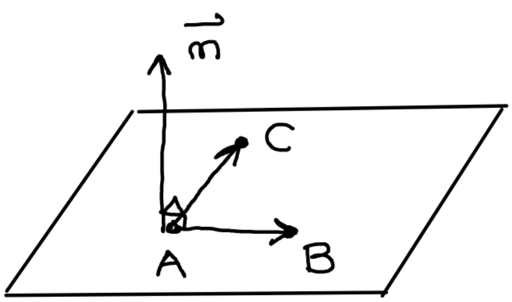
$$ax - ax_0 + by - by_0 + cz - cz_0 = 0$$

$$ax + by + cz = ax_0 + by_0 + cz_0$$

$$\vec{m} \cdot \vec{r}_T = \vec{m} \cdot \vec{r}_{T_0}$$

$$\boxed{\vec{m} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{m} \cdot \vec{r}_{T_0}}$$

enačba ravnine



$$\vec{m} = \overrightarrow{AB} \times \overrightarrow{AC}$$

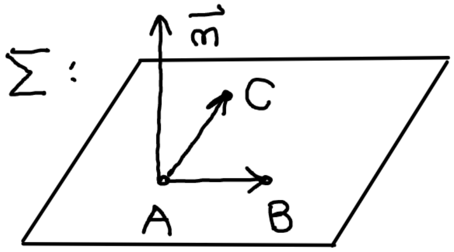
2. Dane so točke $A(2,-3,6)$, $B(-4,-1,6)$, $C(2,-1,9)$ in $D(1,2,-1)$

(a) Poišči enačbo ravnine Σ, ki gre skozi točke A, B in C.

(b) Poišči enačbo premice p, ki gre skozi D in je pravokotna na ravnino Σ.

(c) Izračunaj presečišče premice p in ravnine Σ.

a) Σ:



$$\vec{AB} = \vec{r}_B - \vec{r}_A = \begin{bmatrix} -4 \\ -1 \\ 6 \end{bmatrix} - \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ 0 \end{bmatrix}$$

$$\vec{m} = \vec{AB} \times \vec{AC} =$$

$$\vec{AC} = \vec{r}_C - \vec{r}_A = \begin{bmatrix} 2 \\ -1 \\ 9 \end{bmatrix} - \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \\ -12 \end{bmatrix} =$$

$6 \rightarrow 2\cdot 3 - 2\cdot 0$

$$\begin{matrix} -6 & & 0 \\ 2 & & 2 \end{matrix}$$

$$= 6 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

Enačba ravnine: $\vec{m} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{m} \cdot \vec{r}_A$

$$\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$$

$$x + 3y - 2z = 1\cdot 2 + 3(-3) + (-2)\cdot 6$$

$$\boxed{x + 3y - 2z = -19}$$

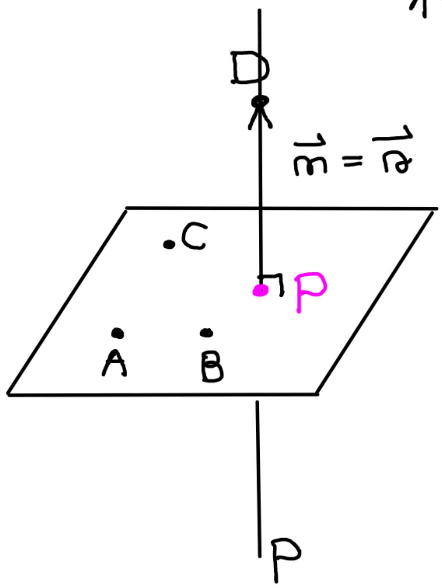
(b) Poišči enačbo premice p, ki gre skozi D in je pravokotna na ravnino Σ.

Ali D leži na Σ? NE

$D(1,2,-1) \rightarrow x + 3y - 2z = -19$

$1 + 3 \cdot 2 - 2(-1) = -19$

$9 = -19$ //



Enačba premice p:

$\vec{r} = \vec{r}_D + t \cdot \vec{m}$, $t \in \mathbb{R}$

$$\boxed{\vec{r} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}}$$

(c) Izračunaj presečišče premice p in ravnine Σ.

P

Enačbo premice preoblikujemo v parametrično obliko:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

$$\downarrow$$

$$\boxed{\begin{aligned} x &= 1 + t \\ y &= 2 + 3t \\ z &= -1 - 2t \end{aligned} \quad t \in \mathbb{R}}$$

Enačbo premice p vstavimo v enačbo ravnine $\Sigma: x + 3y - 2z = -19$:

$$1+t+3(2+3t)-2(-1-2t)=-19$$
$$1+t+6+9t+2+4t=-19$$
$$14t=-28 \quad /:14$$
$$\underline{\underline{t=-2}}$$

Vstavimo dobljen t v enačbo premice p, da dobimo koordinate presečišča P:

$$x=1+t=1-2=-1$$
$$y=2+3t=2+3(-2)=-4$$
$$z=-1-2t=-1-2\cdot(-2)=3$$

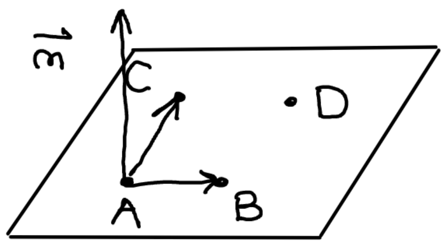
$$P(-1,-4,3)$$

3. Dane so točke $A(1,0,-3)$, $B(-1,0,1)$, $C(3,2,0)$ in $D(4,2,-2)$.

(a) Prepričaj se, da vse štiri ležijo na isti ravnini. Poišči še enačbo te ravnine.

(b) Naj bo p premica, ki gre skozi A in B, q pa premica, ki gre skozi C in D. Zakaj se ti dve premici sekata? Kolikšen je kot med njima?

a)



Enačba ravnine skozi točke A, B, C:

$$\vec{m} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} \times \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -8 \\ 14 \\ -4 \end{bmatrix} = 2 \begin{bmatrix} -4 \\ 7 \\ -2 \end{bmatrix}$$

$$\begin{matrix} -2 & & 2 \\ 0 & & 2 \end{matrix}$$

$$\overrightarrow{AB} = \vec{r}_B - \vec{r}_A = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$$

$$\overrightarrow{AC} = \vec{r}_C - \vec{r}_A = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

Enačba ravnine: $\vec{m} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{m} \cdot \vec{r}_A$

$$\begin{bmatrix} -4 \\ 7 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 7 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

$$-4x + 7y - 2z = (-4)\cdot 1 + 7 \cdot 0 + (-2)(-3)$$

$$\boxed{-4x + 7y - 2z = 2}$$

(4,2,-2)

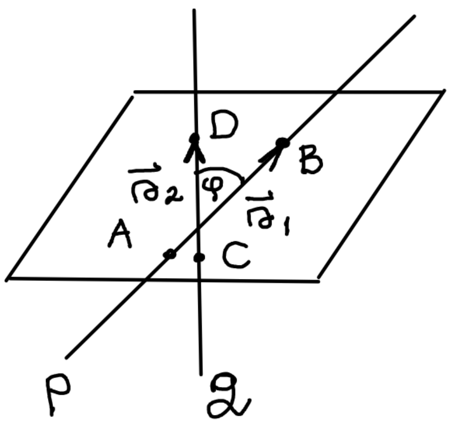
Ali D leži na tej ravnini? DA

D vstavimo v enačbo ravnine:

$$-4\cdot 4+7\cdot 2-2(-2)=2$$

$$2=2 \quad ✓$$

(b) Naj bo p premica, ki gre skozi A in B, q pa premica, ki gre skozi C in D. Zakaj se ti dve premici sekata? Kolikšen je kot med njima?



$\vec{a}_1 = \overrightarrow{AB}$

$\vec{a}_2 = \overrightarrow{CD}$

Premici se sekata, če njuna smerna vektorja nista vzporedna ($\vec{a}_1 \times \vec{a}_2 \neq 0$)

premici

Kot med premicama je kot med njunima smernima vektorjema:

$$\cos\varphi = \frac{\vec{a}_1 \cdot \vec{a}_2}{|\vec{a}_1|\cdot|\vec{a}_2|}$$