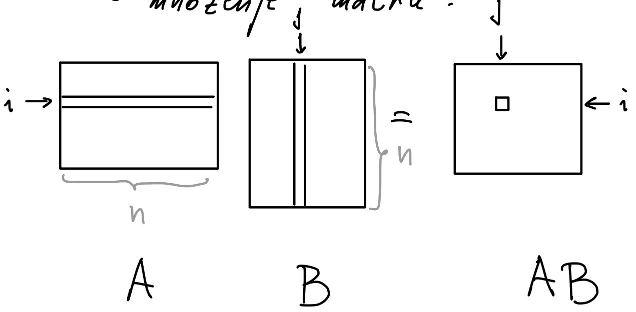
Matematika VSP, vaje, 13.1.2022 ($9^{15}-11^{00}$, P18)

1. Dane so matrike

$$A=\begin{bmatrix}1 & 2 & 3\\ 2 & -4 & 6\end{bmatrix},\ B=\begin{bmatrix}1 & 2\\ 2 & -4\\ 0 & 0\end{bmatrix},\ C=\begin{bmatrix}1 & 1 & 3\\ 5 & 0 & 1\\ 0 & 0 & 1\end{bmatrix}.$$

Izračunaj matriko $(BA)^{\mathsf{T}}+2C^2-I_3$, kjer je I_3 identična matrika velikosti 3×3.

Operacije z matrikami:
- seštevanje po komponentah (seštejemo istoležne elemente)
- množenje matrike s številom (vse elemente v mat. zmnožimo s tem št.)
- množenje matrik :



$$(AB)_{ij}=A_{i1}B_{1j}+A_{i2}B_{2j}+\dots+A_{in}B_{nj}$$

$$BA=\begin{bmatrix}1 & 2\\ 2 & -4\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 2 & -4 & 6\end{bmatrix}=\begin{bmatrix}1\cdot1+2\cdot2, & 1\cdot2+2\cdot(-4), & 1\cdot3+2\cdot6\\ -6 & 20 & -18\\ 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}5 & -6 & 15\\ -6 & 20 & -18\\ 0 & 0 & 0\end{bmatrix}$$

$$C^2=C\cdot C=\begin{bmatrix}1 & 1 & 3\\ 5 & 0 & 1\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 1 & 3\\ 5 & 0 & 1\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}6 & 1 & 7\\ 5 & 5 & 16\\ 0 & 0 & 1\end{bmatrix}$$

- transponiranje : A^T je matrika, v kateri zamenjamo vlogo vrstic in stolpcev mat. A

$$(BA)^T=\begin{bmatrix}5 & -6 & 0\\ -6 & 20 & 0\\ 15 & -18 & 0\end{bmatrix}$$ (Velja $(BA)^T=A^TB^T$.)

Končno $$(BA)^T+2C^2-I_3=\begin{bmatrix}5 & -6 & 0\\ -6 & 20 & 0\\ 15 & -18 & 0\end{bmatrix}+2\cdot\begin{bmatrix}6 & 1 & 7\\ 5 & 5 & 16\\ 0 & 0 & 1\end{bmatrix}-\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}16 & -4 & 14\\ 4 & 29 & 32\\ 15 & -18 & 1\end{bmatrix}.$$

$$I_n=\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & 1\end{bmatrix}$$ $(IA=A,\ BI=B.)$

2. Za naravno število $n \geq 1$ izračunaj:

(a) $\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}^n$, (b) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n$.

(b) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

Zgleda, da je $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$. ← RES VELJA!

Preverimo z mat. indukcijo:

- baza ind., $n=1$: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. ✓
- ind. korak: Če velja za n, potem velja za $n+1$:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n+1 \\ 0 & 1 \end{bmatrix}. \checkmark$$

3. Izračunaj inverze spodnjih matrik:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & -1 & 2 \\ 1 & -1 & 3 \\ -1 & 0 & 1 \end{bmatrix}.$$

Matrika A^{-1} je inverz mat. A, če velja $A^{-1}A = \underline{AA^{-1} = I}$.

Kako poiščemo inverz A?

Označimo $X = A^{-1}$, iščemo je mat. X, da velja $AX = I$.

Pišimo $X = [\vec{x}_1, \vec{x}_2, \vec{x}_3]$, tedaj

$$AX = A[\vec{x}_1, \vec{x}_2, \vec{x}_3] = [A\vec{x}_1, A\vec{x}_2, A\vec{x}_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

torej $A\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $A\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $A\vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Če rešimo te tri sisteme linearnih enačb, dobimo $A^{-1} = [\vec{x}_1, \vec{x}_2, \vec{x}_3]$.

Te sisteme lahko rešujemo hkrati, saj so oper. Gaussove eliminacije odvisne le od A.

$$[A \mid I] \to \text{G.e.} \to [I \mid \vec{x}_1, \vec{x}_2, \vec{x}_3] = [I \mid A^{-1}].$$

V našem primeru:

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{v_1 \leftrightarrow v_2 \\ v_2 \leftarrow v_2 - 2v_1 \\ v_3 \leftarrow v_3 - v_1}]{} \left[\begin{array}{ccc|ccc} \underline{1} & -1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & -2 & 0 \\ 0 & 3 & 1 & 0 & -1 & 1 \end{array}\right] \xrightarrow[\substack{v_2 \leftarrow v_3 - v_2 \\ v_3 \leftarrow v_3 - 3v_2 \\ v_1 \leftarrow v_1 + v_2}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & \underline{1} & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 3 & -4 & -2 \end{array}\right]$$

$$\overbrace{\begin{matrix} -1 & 2 & 1 \\ -1 & 1 & 1 \\ 3 & -4 & -2 \end{matrix}}^{A^{-1}}$$

Torej $A^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ -1 & 1 & 1 \\ 3 & -4 & -2 \end{bmatrix}$. Naredi preizkus: $A^{-1}A = I$,

$AA^{-1} = I$.

Poskusimo poiskat še inverz $C = \begin{bmatrix}0 & -1 & 4\\ 1 & -1 & 3\\ -1 & 0 & 1\end{bmatrix}$:

$$[C\,|\,I] = \left[\begin{array}{ccc|ccc}0 & -1 & 4 & 1 & 0 & 0\\ 1 & -1 & 3 & 0 & 1 & 0\\ -1 & 0 & 1 & 0 & 0 & 1\end{array}\right] \xrightarrow[\substack{v_1 \leftrightarrow v_2\\ v_3 \leftarrow v_3 + v_1}]{} \left[\begin{array}{ccc|ccc}\underline{1} & -1 & 3 & 0 & 1 & 0\\ 0 & -1 & 4 & 1 & 0 & 0\\ 0 & -1 & 4 & 0 & 1 & 1\end{array}\right] \longrightarrow$$

$$\xrightarrow[\substack{v_2 \leftarrow v_2 \cdot (-1)\\ v_3 \leftarrow v_3 + v_2}]{} \left[\begin{array}{ccc|ccc}1 & -1 & 3 & 0 & 1 & 0\\ 0 & \underline{1} & -4 & -1 & 0 & 0\\ 0 & 0 & 0 & -1 & 1 & 1\end{array}\right], \text{ tj. } 0 = -1,\ 0 = 1 \text{ in } 0 = 1.$$

To je protislovje, pripadajoč sistem nima rešitev, ta matrika nima inverza.

4. Reši matrične enačbe z neznanimi matrikami X, Y, Z.

$$\text{(a) } \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} X = \begin{bmatrix}3 & 5\\ 5 & 9\end{bmatrix}, \text{ (b) } Y \begin{bmatrix}3 & -2\\ 5 & -4\end{bmatrix} = \begin{bmatrix}-1 & 2\\ -5 & 6\end{bmatrix}, \text{ (c) } \begin{bmatrix}3 & -1\\ 5 & -2\end{bmatrix} Z \begin{bmatrix}5 & 6\\ 7 & 8\end{bmatrix} = \begin{bmatrix}14 & 16\\ 9 & 10\end{bmatrix}.$$

(a) $AX = B$

Naivno $X = \begin{bmatrix}x_1 & x_2\\ x_3 & x_4\end{bmatrix}$, tedaj $AX = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}x_1 & x_2\\ x_3 & x_4\end{bmatrix} = \begin{bmatrix}x_1 + 2x_3, & x_2 + 2x_4\\ 3x_1 + 4x_3, & 3x_2 + 4x_4\end{bmatrix} = \begin{bmatrix}3 & 5\\ 5 & 9\end{bmatrix}$,

ima 2 vrstici
in 2 stolpca

torej:
$$\boxed{\begin{array}{lllll} x_1 & & +2x_3 & & = 3\\ & x_2 & & +2x_4 & = 5\\ 3x_1 & & +4x_3 & & = 5\\ & 3x_2 & & 4x_4 & = 9\end{array}}$$

Reši samostojno!

Bolj učinkovito:

$X = [\vec{x}_1, \vec{x}_2]$ ... $AX = [A\vec{x}_1, A\vec{x}_2] = [\vec{b}_1, \vec{b}_2] = B$

$[A\,|\,B] \rightarrow$ G.e. $\rightarrow [I\,|\,X]$

$$[A\,|\,B] = \left[\begin{array}{cc|cc}\underline{1} & 2 & 3 & 5\\ 3 & 4 & 5 & 9\end{array}\right] \rightarrow \left[\begin{array}{cc|cc}1 & 2 & 3 & 5\\ 0 & -2 & -4 & -6\end{array}\right] \rightarrow \left[\begin{array}{cc|cc}1 & 0 & -1 & -1\\ 0 & 1 & 2 & 3\end{array}\right], \quad X = \begin{bmatrix}-1 & -1\\ 2 & 3\end{bmatrix}$$

Še drugače: $A^{-1}\cdot /\ AX = B$ ... $\underbrace{A^{-1}A}_{I} X = A^{-1}B$ ... $\underline{X = A^{-1}B}$.

(b) $YA = B \;/\; A^{-1} \;\ldots\; Y = BA^{-1}$

$$[A|I] = \left[\begin{array}{cc|cc} 3 & -2 & 1 & 0 \\ 5 & -4 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & -2/3 & 1/3 & 0 \\ 0 & -2/3 & -5/3 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & \frac{5}{2} & -\frac{3}{2} \end{array}\right],$$

torej $A^{-1} = \begin{bmatrix} 2 & -1 \\ \frac{5}{2} & -\frac{3}{2} \end{bmatrix}$.

Zato $Y = BA^{-1} = \begin{bmatrix} -1 & 2 \\ -5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ \frac{5}{2} & -\frac{3}{2} \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 5 & -4 \end{bmatrix}$.

Drugače: $YA = B \;/^{T} \;\ldots\; (YA)^T = B^T \;\ldots\; A^T Y^T = B^T$

$$[A^T \,|\, B^T] \to \text{G.e.} \to [I \,|\, Y^T]$$

$$[A^T|B^T] = \left[\begin{array}{cc|cc} 3 & 5 & -1 & -5 \\ -2 & -4 & 2 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 0 & -2 & 4 & 8 \end{array}\right] \longrightarrow \left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \;\middle|\; \overbrace{\begin{matrix} 3 & 5 \\ -2 & -4 \end{matrix}}^{Y^T} \right],$$

kar pomeni $Y = \begin{bmatrix} 3 & -2 \\ 5 & -4 \end{bmatrix}$.