Mathematical Modelling Exam

June 28th, 2022

This is an open book exam. You are allowed to use your notes, books and any other literature. You are NOT allowed to use any communication device. You have 100 minutes to solve the problems.

1. Let

$$A = \begin{bmatrix} 0 & 2 \\ -2 & -1 \\ -2 & 0 \end{bmatrix}$$

be a matrix.

(a) Compute the truncated singular value decomposition of A.

(b) Does there exist a matrix $B \in \mathbb{R}^{3\times 2}$ of rank 1 such that $\|A - B\|_F = 1$? If yes, compute it, otherwise justify, why it does not exist.

Solution.

(a) We have that

$$A^T A = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix},$$

which implies

$$\det(A^T A - \lambda I_2) = (8-\lambda)(5-\lambda) - 2^2 = \lambda^2 - 13\lambda + 36 = (\lambda - 9)(\lambda - 4).$$

So the eigenvalues of $A^T A$ are $\lambda_1 = 9$, $\lambda_2 = 4$. Hence, Σ in the SVD of $A = U\Sigma V^T$ is equal to

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

The kernel of

$$A^T A - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$$

contains the vector $\begin{bmatrix} 2 & 1 \end{bmatrix}^T$ and hence

$$v_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Further on,

$$u_1 = \frac{1}{3} A v_1 = \frac{1}{3\sqrt{5}} \begin{bmatrix} 2 \\ -5 \\ -4 \end{bmatrix}.$$

The kernel of

$$A^T A - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

contains the vector $\begin{bmatrix} 1 & -2 \end{bmatrix}^T$ and hence

$$v_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Further on,

$$u_2 = \frac{1}{2} A v_2 = \frac{1}{2\sqrt{5}} \begin{bmatrix} -4 \\ 0 \\ -2 \end{bmatrix}.$$

So, the truncated SVD of A is equal to

$$A = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T.$$

(b) By the Eckart-Young theorem the matrix B of rank 1, which minimizes the norm $\|A-B\|_F$ is equal to $\sigma_1 u_1 v_1^2$. The distance $\|A-B\|_F$ is $\|\sigma_2 u_2 v_2^T\|_F = \sigma_2 = 2$. Hence, there does not exist a matrix B of rank 1 satisfying $\|A-B\|_F = 1$.

2. Let $f(x, y, z) = x^2 + 3xy + yz^3$ be a function of three variables.

(a) Compute the gradient ∇f.

(b) Perform one step of Newton's method to approximate the stationary point of f using the initial approximation $(x_0, y_0, z_0) = \left(1, 0, \frac{1}{\sqrt{3}}\right)$.

Solution.

(a)

$$\nabla f(x, y, z) = \begin{bmatrix} \frac{\partial f}{\partial x}(x, y, z) \\ \frac{\partial f}{\partial y}(x, y, z) \\ \frac{\partial f}{\partial z}(x, y, z) \end{bmatrix} = \begin{bmatrix} 2x + 3y \\ 3x + z^3 \\ 3yz^2 \end{bmatrix}.$$

(b) We are searching for the solution of $\nabla f(x, y, z) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ using Newton's method. We have that

$$v^{(1)} = v^{(0)} - \left(J(\nabla f)(v^{(0)})\right)^{-1} (\nabla f)(v^{(0)}),$$

where

$$J(\nabla f)(x, y, z) = \begin{bmatrix} \frac{\partial^2 f(x,y,z)}{\partial x^2} & \frac{\partial^2 f(x,y,z)}{\partial x \partial y} & \frac{\partial f(x,y,z)}{\partial x \partial z} \\ \frac{\partial^2 f(x,y,z)}{\partial x \partial y} & \frac{\partial^2 f(x,y,z)}{\partial y^2} & \frac{\partial f(x,y,z)}{\partial y \partial z} \\ \frac{\partial^2 f(x,y,z)}{\partial x \partial x} & \frac{\partial^2 f(x,y,z)}{\partial y \partial z} & \frac{\partial f(x,y,z)}{\partial z^2} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 0 & 3z^2 \\ 0 & 3z^2 & 6yz \end{bmatrix}$$

is the Jacobian matrix of ∇f. So

$$J(\nabla f)(v^{(0)}) = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

We compute $\left(J(\nabla f)(v^{(0)})\right)^{-1}$ using Gaussian elimination:

$$\underbrace{\left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]}_{\left[\ J(\nabla f)(v^{(0)})\ \middle|\ I_3\ \right]} \underbrace{\sim}_{\ell_2 \leftrightarrow \ell_3} \left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\underbrace{\sim}_{\ell_1 = \frac{1}{2}(\ell_1 - 3\ell_2)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\underbrace{\sim}_{\ell_3 = \ell_3 - 3\ell_1} \underbrace{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{3}{2} & 1 & \frac{9}{2} \end{array}\right]}_{\left[\ I_3\ \middle|\ \left(J(\nabla f)(v^{(0)})\right)^{-1}\ \right]}.$$

So

$$v^{(1)} = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{\sqrt{3}} \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & 0 & -\frac{3}{2} \\ 0 & 0 & 1 \\ -\frac{3}{2} & 1 & \frac{9}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 3 + \frac{1}{3\sqrt{3}} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{2}{3\sqrt{3}} \end{bmatrix}.$$

3. Let

$$f(t) = (t^3 - 5t^2 + 3t + 11, t^2 - 2t + 3), \quad t \in \mathbb{R}$$

be the parametric curve.

(a) Find all points on the curve, where the tangent is horizontal or vertical.

(b) Find all self-intersections.

(c) Sketch the curve.

Solution.

(a) The tangent is horizontal in the points where $y'(t) = 0$:

$$2t - 2 = 0 \quad \Leftrightarrow \quad t = 1.$$

The corresponding point is

$$f(1) = (1 - 5 + 3 + 11, 1 - 2 + 3) = (10, 2).$$

The tangent is horizontal in the points where $x'(t) = 0$:

$$3t^2 - 10t + 3 = 0 \quad \Leftrightarrow \quad t_{1,2} = \frac{10 \pm \sqrt{100 - 36}}{6} \in \left\{3, \frac{1}{3}\right\}.$$

The corresponding points are

$$\begin{aligned} f(3) &= (27 - 45 + 9 + 11, 9 - 6 + 3) = (2, 6), \\ f(3^{-1}) &= (3^{-3} - 5 \cdot 3^{-2} + 1 + 11, 3^{-2} - 2 \cdot 3^{-1} + 3) = \left(\frac{310}{27}, \frac{22}{9}\right) \approx (11.5, 2.4). \end{aligned}$$

(b) The curve has self-intersections, where $f(t) = f(s)$ for $t \neq s$. We have that

$$\begin{aligned} & t^3 - 5t^2 + 3t + 11 = s^3 - 5s^2 + 3s + 11 \\ \Leftrightarrow \quad & t^3 - s^3 = 5(t^2 - s^2) - 3(t - s) \\ \Leftrightarrow \quad & t^2 + ts + s^2 = 5(t + s) - 3, \end{aligned} \tag{1}$$

where we divided by $t - s$ in the last line. Further on,

$$\begin{aligned} & t^2 - 2t + 3 = s^2 - 2s + 3 \\ \Leftrightarrow \quad & t^2 - s^2 = 2(t - s) \\ \Leftrightarrow \quad & t + s = 2, \end{aligned} \tag{2}$$

where we divided by $t - s$ in the last line. We use (2) in (1):

$$t^2 + t(2 - t) + (2 - t)^2 = 10 - 3 = 7$$

and hence

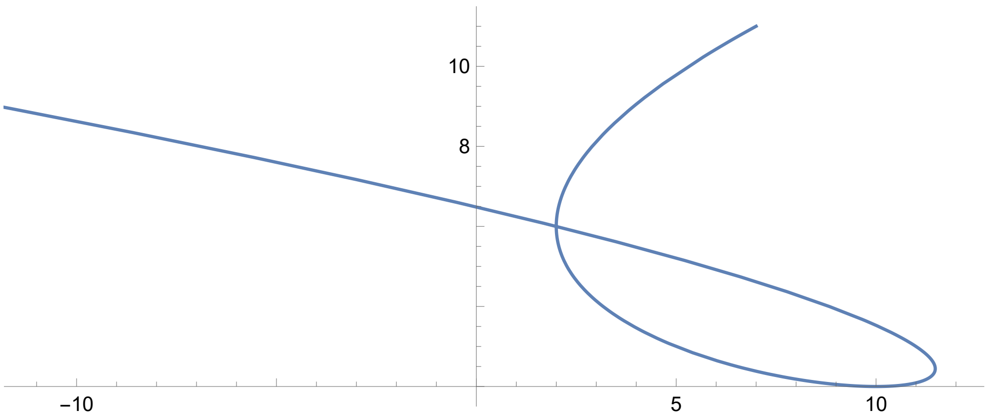
$$0 = t^2 - 2t - 3 = (t - 3)(t + 1).$$

The solutions are $t_1 = 3$, $t_2 = -1$ with the correspoding $s_1 = -1$ and $s_2 = 3$. So the only point of self-intersection is $f(-1) = f(3) = (2, 6)$.

(c) We compute

$$\lim_{t \to -\infty} f(t) = (-\infty, \infty) \qquad \text{and} \qquad \lim_{t \to \infty} f(t) = (\infty, \infty).$$

The sketch of the curve is the following:



4. Let

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix},$$

where $A \in \mathbb{R}^{3 \times 3}$, be a system of differential equations with the following three solutions:

$$e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad e^{t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad e^{2t} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

(a) Write down a general solution of the system.

(b) Determine the matrix A.

(c) Write down a third order differential equation with constants coefficients, which is transformed into the above system.

Solution.

(a) A general solution of the system is

$$C_1e^{-t}\begin{bmatrix}1\\-1\\1\end{bmatrix}+C_2e^{t}\begin{bmatrix}1\\1\\1\end{bmatrix}+C_3e^{2t}\begin{bmatrix}1\\2\\4\end{bmatrix},$$

where C_1, C_2, C_3 are constants.

(b) The matrix A has eigenpairs $(-1, v_1)$, $(1, v_2)$, $(2, v_3)$, where

$$v_1=\begin{bmatrix}1\\-1\\1\end{bmatrix},\quad v_2=\begin{bmatrix}1\\1\\1\end{bmatrix},\quad v_3=\begin{bmatrix}1\\2\\4\end{bmatrix}.$$

Hence,

$$A=\underbrace{\begin{bmatrix}v_1 & v_2 & v_3\end{bmatrix}}_{P}\cdot\operatorname{diag}(-1,1,-2)\cdot P^{-1}.$$

Let us compute P^{-1} using Guassian elimination:

$$\underbrace{\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\-1&1&2&0&1&0\\1&1&4&0&0&1\end{array}\right]}_{[\,P\,|\,I_3\,]}\underbrace{\sim}_{\substack{\ell_2=\ell_2+\ell_1\\ \ell_3=\ell_3-\ell_1}}\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&2&3&1&1&0\\0&0&3&-1&0&1\end{array}\right]$$

$$\underbrace{\sim}_{\substack{\ell_1=\ell_1-\frac{1}{3}\ell_3\\ \ell_2=\ell_2-\ell_3}}\left[\begin{array}{ccc|ccc}1&1&0&\frac{4}{3}&0&-\frac{1}{3}\\0&2&0&2&1&-1\\0&0&3&-1&0&1\end{array}\right]$$

$$\underbrace{\sim}_{\substack{\ell_1=\ell_1-\frac{1}{2}\ell_2\\ \ell_2=\frac{1}{2}\ell_2\\ \ell_3=\frac{1}{3}\ell_3}}\underbrace{\left[\begin{array}{ccc|ccc}1&0&0&\frac{1}{3}&-\frac{1}{2}&\frac{1}{6}\\0&1&0&1&\frac{1}{2}&-\frac{1}{2}\\0&0&1&-\frac{1}{3}&0&\frac{1}{3}\end{array}\right]}_{[\,I_3\,|\,P^{-1}\,]}.$$

So

$$\begin{aligned}A&=\begin{bmatrix}1&1&1\\-1&1&2\\1&1&4\end{bmatrix}\begin{bmatrix}-1&0&0\\0&1&0\\0&0&2\end{bmatrix}\begin{bmatrix}\frac{1}{3}&-\frac{1}{2}&\frac{1}{6}\\1&\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{3}&0&\frac{1}{3}\end{bmatrix}\\&=\begin{bmatrix}-1&1&2\\1&1&4\\-1&1&8\end{bmatrix}\begin{bmatrix}\frac{1}{3}&-\frac{1}{2}&\frac{1}{6}\\1&\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{3}&0&\frac{1}{3}\end{bmatrix}\\&=\begin{bmatrix}0&1&0\\0&0&1\\-3&0&2\end{bmatrix}.\end{aligned}$$

(c) Since eigenvalues of the matrix A are $-1, 1, 2$, the corresponding third order polynomial is

$$(\lambda+1)(\lambda-1)(\lambda-2) = (\lambda^2-1)(\lambda-2) = \lambda^3 - 2\lambda^2 - \lambda + 2$$

and hence the differential equation with constants coefficients is

$$x^{(3)} = 2x^{(2)} + x^{(2)} - 2x.$$