Homology: definition and computation

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Now that we presented combinatorial and algebraic prerequisites, we are ready to define homology. The notion of homology arose from the need to detect the holes in a simplicial complex or a more general space. Its definition is not as straight forward as one might hope, but nonetheless results in a notion amenable to practical computations and consistent with the geometric intuition we presented in the first chapter.

In this chapter we will journey through a geometric introduction and definition of homology, and study the basic methods of computation. We will provide examples of homologies, which should build up our understanding and detection of holes of all dimension not only in Euclidean spaces, but also within the combinatorial context of abstract simplicial complexes.

1 Definition

Homology measures holes in simplicial complex. As the later is provided by a collection of simplices, we need to devise a computational framework based on the simplices that will result in a meaningful result. The formal treatment of this section will be provided in parallel to a simple example on the right.

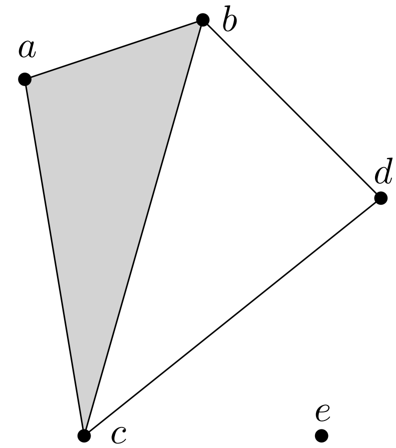

Figure 1: Abstract simplicial complex L.

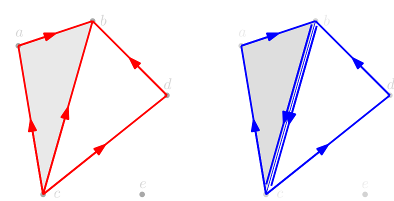

Figure 2: Two 1-chains in L: the red chain on the left $\langle c,a\rangle + \langle a,b\rangle + \langle c,d\rangle + \langle d,b\rangle + \langle c,b\rangle$ coincides with the blue chain on the right $\langle c,a\rangle + \langle a,b\rangle + \langle c,d\rangle + \langle d,b\rangle - 2\langle c,b\rangle = \langle c,a\rangle + \langle a,b\rangle + \langle c,d\rangle + \langle d,b\rangle + 2\langle b,c\rangle$ iff the coefficients are from $\mathbb{Z}_3$.

Let K be an abstract simplicial complex of dimension n and choose a field of coefficients $\mathbb{F}$.

Chains

Chains are formal sums of simplices along with coefficients from $\mathbb{F}$. They are an algebraic model of collections of simplices as demonstrated in Figure 3.

For each $p \in \{0, 1, \dots, n\}$ let n_p denote the number of simplices of dimension p in K.

Definition 1.1. *A p-**chain** is a formal sum $\sum_{i=1}^{n_p} \lambda_i \sigma_i^p$ with $\lambda_i \in \mathbb{F}$ and σ_i^p being an oriented simplex of dimension p in K for each i.*

This formalism incorporates the signatures of orientation: if σ is an oriented simplex then $(-1) \cdot \sigma = -\sigma$ is the simplex σ with the changed orientation.

We assume that $\{\sigma_2^p, \sigma_2^p, \dots, \sigma_{n_p}^p\}$ is the collection of all p-simplices of K, with the p-simplices that are "absent" having coefficient 0. p-

chains can be added/subtracted and multiplied by any scalar:

$$\sum_{i=1}^{n_p} \lambda_i \sigma_i^p + \sum_{i=1}^{n_p} \lambda_i' \sigma_i^p = \sum_{i=1}^{n_p} (\lambda_i + \lambda_i')\sigma_i^p \qquad \forall \lambda_i, \lambda_i' \in \mathbb{F}.$$

$$k \sum_{i=1}^{n_p} \lambda_i \sigma_i^p = \sum_{i=1}^{n_p} (k\lambda_i)\sigma_i^p, \qquad \forall k, \lambda_i \in \mathbb{F}.$$

Example 1.2. *Consider simplicial complex L from Figure 1. Two examples of* 1*-chains and their additions are presented in Figure 3.*

- *Working in* $\mathbb{Z}_2$ *(top of Figure 3) the* 1*-chains are merely subsets of the collection of edges as the orientation does not matter* ($+1 = -1$ *in* $\mathbb{Z}_2$*). Adding the red chain* $\{a, c\} + \{b, c\}$ *and the blue chain* $\{b, c\} + \{b, d\}$ *results in the purple chain* $\{a, c\} + \{b, d\}$*.*
- *Computing in any other field (bottom of Figure 3) the orientation does matter. Adding the red chain* $\langle b, a\rangle + \langle a, c\rangle + \langle c, d\rangle$ *and the blue chain* $\langle b, a\rangle + \langle d, c\rangle$ *results in the purple chain* $\langle a, c\rangle + 2\langle b, a\rangle$*.*

As a result the collection of all chains forms a vector space[1].

Definition 1.3. *The* ***chain group*** $C_p(K; \mathbb{F})$ *is the vector space of all* p*-chains.*

Thinking of p-simplices of K as an abstract collection of linearly independent vectors, the resulting linear space (with coefficients in $\mathbb{F}$) spanned by them is the chain group. If n_p is the number of p-simplices of K then $C_p(K; \mathbb{F}) \cong \mathbb{F}^{n_p}$.

Boundary

With the definition of chain groups in place, we can now express the boundary relation as a linear map. The boundary map encodes the assembly instruction for a simplicial complex.

Definition 1.4. *Let* $p \in \mathbb{N}$*. The* ***boundary map***

$$\partial_p \colon C_p(K; \mathbb{F}) \to C_{p-1}(K; \mathbb{F})$$

is the linear map defined by the following rule on the basis of $C_p(G; \mathbb{F})$*: for each oriented* p*-simplex* $\sigma = \langle v_0, v_1, \ldots, v_p\rangle$ *the image* $\partial_p \sigma$ *is the sum of facets of* σ *equipped with the induced orientation from* σ*, i.e.,:*

$$\partial_p \sigma = \sum_{i=0}^{p} (-1)^i \langle v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k p\rangle.$$

For technical reasons we additionally define $\partial_0 \colon C_0(K; \mathbb{F}) \to 0$ *to be*

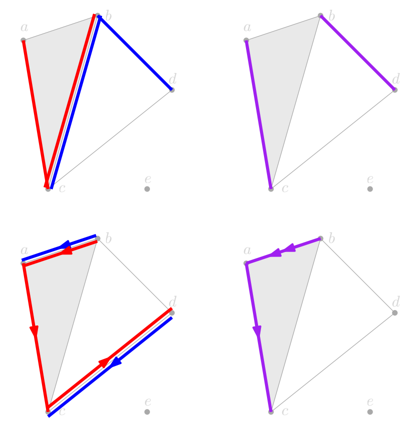

Figure 3: Top row: addition of chains in $\mathbb{Z}_2$. Bottom row: addition of chains in any other field.

[1] For historical and practical reasons we will match the established terminology in the literature and call this vector space a chain group, the reason being that if the coefficients are in a group (as is standard in classical theoretical approaches, see also an Appendix), the resulting chains form only a group. In our case the chains still form a group for addition, but the overall structure along with the multiplication by a scalar is that of a vector space.

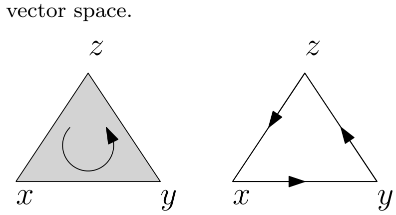

Figure 4: Oriented triangle $\langle x, y, z\rangle$ and its boundary $\partial_2(\langle x, y, z\rangle) = \langle x, y\rangle + \langle y, z\rangle + \langle z, x\rangle$.

⚠ We will typically be dropping the index of the boundary map ∂ whenever it will be evident either that the statement relating to the use of ∂ refers to all indices p or to a specific p. For example, when talking about $\partial\sigma^p$, it is apparent that the map in question is ∂_p. On the other hand, notation $\partial^2 = 0$ means that for each $p \in \mathbb{N}, \partial_p \circ \partial_{p-1}$ is the trivial map whose image is the zero vector.

the trivial map (actually, the only map) into the trivial vector space (the space only containing the 0 *vector).*

A crucial fact for the algebraic formulation of a homology theory is that the composition of two boundary maps is the trivial map. In particular, this implies that the image of a boundary map is contained in the kernel of the subsequent boundary map. See the note on the right concerning the notation in the following statement.

Theorem 1.5. $\partial^2 = 0$.

Proof. It suffices to prove that $\partial^2\sigma = 0$ for an oriented p-simplex $\sigma = \langle v_0, v_1, \ldots, v_p \rangle$. Note that $\partial^2\sigma$ is a formal sum of faces of σ of dimension $p-2$. Choose indices $i < j$ from $\{0, 1, \ldots, p\}$ and consider how does the face[2]

$$\sigma' = \langle v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, v_{j+1}, \ldots, v_p \rangle$$

appear in $\partial^2\sigma$. Such a face appears from two terms:

- By first removing vertex v_j from σ in the expression of ∂_p and then removing vertex v_i from the resulting simplez in the expression of ∂_{p-1}. The indices of removed vertices are j and i hence the sign in from t of σ' is $(-1)^i(-1)^j$.
- By first removing vertex v_i from σ in the expression of ∂_p and then removing vertex v_j from the resulting simplez in the expression of ∂_{p-1}. The indices of removed vertices are i and[3] $(j-1)$ hence the sign in from t of σ' is $(-1)^i(-1)^{j-1}$.

As the signs are the opposite, the total sum equals zero. □

[2] The following face is obtained from σ by dropping vertices v_i and v_j.

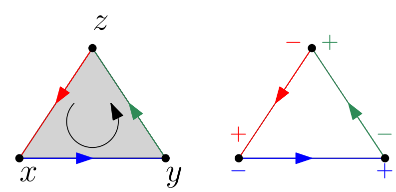

Figure 5: An example of Theorem 1.5: Oriented triangle $\langle x,y,z \rangle$ on the left, its boundary $\partial(\langle x,y,z \rangle) = \langle x,y \rangle + \langle y,z \rangle + \langle z,x \rangle$, and $\partial^2(\langle x,y,z \rangle) = \langle x \rangle - \langle x \rangle + \langle y \rangle - \langle y \rangle + \langle z \rangle - \langle z \rangle = 0$ as indicated by the signs at the vertices on the right.

[3] As vertex v_i has already been removed and $i < j$, the vertex v_j in now on position $j-1$.

Corollary 1.6. $\mathrm{Im}(\partial) \subset \ker(\partial)$.

Definition 1.7. *The collection of chain groups bound together by the boundary maps is called the* ***chain complex:***

$$\cdots \xrightarrow{\partial} C_n(K;\mathbb{F}) \xrightarrow{\partial} C_{n-1}(K;\mathbb{F}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_1(K;\mathbb{F}) \xrightarrow{\partial} C_0(K;\mathbb{F}) \xrightarrow{\partial} 0$$

For computational purposes the boundary maps are typically represented as matrices with entries in $\mathbb{F}$. For each $p \in \mathbb{N}$ a matrix M_p corresponding to ∂_p is obtained as follows:

- Columns are enumerated by oriented p-simplices of K.
- Rows are enumerated by oriented $(p-1)$-simplices of K.

- Entry at position (i, j) equals $+1$ or -1 if the i-th row appears with orientation $+1$ or -1 correspondingly in the boundary of the j-th column. All other entries are zero.

In particular, the boundary $\partial\alpha$ of a chain α is obtained by multiplying the boundary matrix with the natural representation of α in the chosen[4] basis.

[4] The same basis that is used to enumerate rows.

Example 1.8. *Labeled boundary matrices*[5] *for complex L of Figure 6:*

$$M_2 = \begin{array}{c} \\ \langle a,b\rangle \\ \langle b,c\rangle \\ \langle a,c\rangle \\ \langle b,d\rangle \\ \langle c,d\rangle \end{array} \begin{array}{c} \langle a,b,c\rangle \\ \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \end{array}, \quad M_1 = \begin{array}{c|ccccc} & \langle a,b\rangle & \langle b,c\rangle & \langle a,c\rangle & \langle b,d\rangle & \langle c,d\rangle \\ \hline \langle a\rangle & -1 & & -1 & & \\ \langle b\rangle & 1 & -1 & & -1 & \\ \langle c\rangle & & 1 & 1 & & -1 \\ \langle d\rangle & & & & 1 & 1 \\ \langle e\rangle & & & & & \end{array}$$

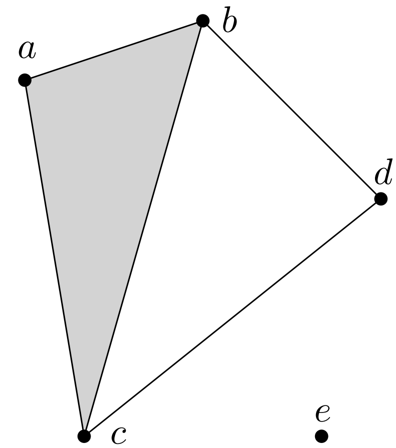

Figure 6: Abstract simplicial complex L.

[5] Only non-zero entries are provided. Matrix M_0 has no formal rows as it represents the zero-map into the one-element vector space 0.

Homology

We are now finally ready to define homology as a measure of holes. Let us first build an intuition on simplicial complex L from Figure 6. This will be followed up by a formal introduction in Definition 1.9.

Our task is to compute that L has one hole. In the figure the hole seems to be enclosed by edges cd, db and bc. Following this observation we decide that holes will be represented by a special kind of chains called cycles, see Figure 7. These are the chains that model closed simplicial loops in our simplicial complex, just as the one describing the hole in L above. Formally, we define cycles to be those chains, whose boundary is zero. These are our candidates for the representatives of holes.

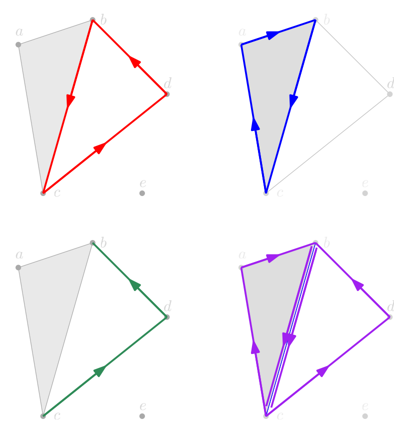
Figure 7: Top row: Two cycles. Bottom row: a chain that is not a cycle (left) and the cycle, that is the sum of the cycles of the top row.

However, not all cycles represent loops. For example, the top right cycle in Figure 7 is a boundary of a triangle and thus does not enclose any hole. Such cycles thus do not represent a hole and should be treated as trivial. Similarly, if a cycle is obtained as the boundary of a 2-chain, then it should be treated as trivial. Such cycles are called boundaries[6] and the structure formalizing the triviality of boundaries is the quotient space.

[6] At this point, the term "boundary" can refer to a geometric boundary of a simplex, a boundary map, or a chain, that is the image of a boundary map.

Summing up the idea, the holes are represented by the quotient group cycles/boundaries.

Recall that for each $p \in \{0, 1, \ldots\}$ we have $\operatorname{Im}\partial_{p+1} \leq \ker\partial_p$.

Definition 1.9. *Let K be an abstract simplicial complex. Choose a field $\mathbb{F}$ and $q \in \{0, 1, \ldots\}$. We define*

- *q-**cycles** as $Z_q(K; \mathbb{F}) = \ker\partial_q \leq C_q(K; \mathbb{F})$.*

- *q-**boundaries** as $B_q(K;\mathbb{F}) = \operatorname{Im} \partial_{q+1} \le Z_q(K;\mathbb{F}) \le C_q(K;\mathbb{F})$.*
- *q-**homology group** as the quotient $H_q(K;\mathbb{F}) = Z_q(K;\mathbb{F})/B_q(K;\mathbb{F})$.*

*The dimension of $H_q(K;\mathbb{F})$ is called the q-**Betti number** (of K with coefficients in $\mathbb{F}$) and is denoted by $\mathfrak{b}_q = \mathfrak{b}_q(K;\mathbb{F})$.*

In particular, each element of a homology group is an equivalence class[7] of cycles. The homology group of example L from Figure 6 will depend on $\mathbb{F}$. Defining $\alpha = \langle b,c\rangle + \langle c,d\rangle + \langle d,b\rangle$ as the top left cycle in Figure 7, we see that $H_1(L;\mathbb{F})$ is $\{k[\alpha] \mid k \in \mathbb{F}\}$. Even though we have only one hole, the homology group typically has more elements. However, the entire $H_1(L;\mathbb{F})$ is spanned by $[\alpha]$ and thus the number of holes should be interpreted[8] as the dimension of the homology group, in this case 1.

[7] Given a cycle β, the corresponding class in homology will be denoted by $[\beta]$.

[8] At this point we observe that it is crucial to preserve the algebraic structure (of a vector space) of the homology group in order to compute the dimension as the number of holes.

More generally, each homology group with coefficients in a field $\mathbb{F}$ is a vector space and thus isomorphic to $\mathbb{F}^r$ for some dimension r. The main goal of our computations is thus to compute $r = \mathfrak{b}_q$, which represents the number of q-dimensional holes:

- $\mathfrak{b}_0$ equals for all fields $\mathbb{F}$ and coincides with the number of components (0-dimensional holes).
- $\mathfrak{b}_1$ is the number of holes in the usual geometric sense (1-dimensional holes), although various fields detect different[9] holes in this setting. For planar graphs however, $\mathfrak{b}_1$ is always the number of the holes.
- $\mathfrak{b}_2$ is the number of caves/enclosures.

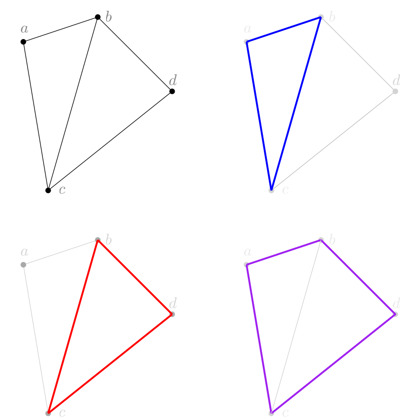

Figure 8: Top left: a simplicial complex with two holes. Its first homology group H_1 with coefficients in $\mathbb{Z}_2$ has three non-trivial elements, depicted as the blue, the red, and the purple chain. However, that does not mean that the number of holes equals 3. Along with the trivial homology class, the homology groups consists of 4 elements. This means that its dimension over $\mathbb{Z}_2$ equals 2, which is the number of holes. We also observe that any two of the three non-trivial chains above could form the basis of H_1. In fact, each of the three non-trivial chains is the sum of the other two.

These interpretations will be explored, demonstrated and partially proved throughout the rest of this chapter. Before we do that let us mention that homology groups are homotopy invariant even though cycles and boundaries are not.

Theorem 1.10. *Assume K and K' are simplicial complexes. Then a homotopy equivalence $K \simeq K'$ implies $H_p(K;\mathbb{F}) \cong H_p(K';\mathbb{F})$, for each field $\mathbb{F}$ and for each $p \in \{0,1,\ldots\}$.*

Zero-dimensional homology

In this subsection we prove that $\mathfrak{b}_0$ is the number of components[10] of the underlying simplicial complex. Let K be a simplicial complex and $\mathbb{F}$ any field. The homology group $H_0(K;\mathbb{F})$ is computed from the following piece of information:

$$C_1(K;\mathbb{F}) \xrightarrow{\partial_1} C_0(K;\mathbb{F}) \xrightarrow{\partial_0} 0.$$

[9] See the example of the Klein bottle later in this section.

[10] While there are alternative ways to obtain the number of components employing a smaller amount of algebra, there are no alternatives to homological constructions when it comes to 1- and higher-dimensional holes.

In order to compute $H_0(K;\mathbb{F})$ we need to determine $\ker\partial_0$ and $\operatorname{Im}\partial_1$. Since ∂_0 is trivial we have[11] $\ker\partial_0 = C_0(K;\mathbb{F})$. In order to determine $\operatorname{Im}\partial_1$ we prove the following proposition.

[11] Dimension 0 is the only case where a single simplex forms a cycle.

Proposition 1.11. *Let K be a simplicial complex, $\mathbb{F}$ any field and assume $x, y \in K^{(0)}$ are vertices. Then $\langle y\rangle - \langle x\rangle \in \operatorname{Im}\partial_1$ iff x and y lie in the same component of K.*

Proof. Assume x and y lie in the same component of K. Then there exists[12] a path from x to y tracing edges. Let $x = x_0, x_1, \dots, x_k = y$ denote the sequence of vertices traced by one such path. Then the chain $\langle y\rangle - \langle x\rangle$ is the boundary of the 1-chain $\sum_{i=0}^{k-1}\langle x_i, x_{i+1}\rangle$. See Figure 9 for an example.

[12] ...by the simplicial approximation Theorem.

In order to prove the other direction assume $\langle y\rangle - \langle x\rangle = \partial\alpha$ for some 1-chain α. Let $K' \leq K$ be the component of K containing vertex x and define α' to be the part of α contained in K'. i.e., σ' contains all those terms of α whose edge is in K'. No vertex of $K'^{(0)} \setminus \{x, y\}$ appears in $\partial\alpha'$ as none appears in $\partial\alpha$ either and the terms containing edges with such a vertex as an endpoint are the same in both α and α'. Hence $\partial\alpha'$ is either $\langle y\rangle - \langle x\rangle$ in case $y \in K'$ or $-\langle x\rangle$ otherwise. Since the coefficients in front of vertices of any boundary add[13] up to zero, only the first of these two options is possible. $\square$

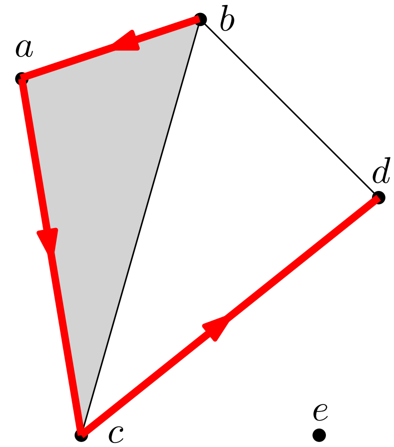

Figure 9: The boundary of the depicted chain is $\langle d\rangle - \langle b\rangle$, which is also the boundary of $\langle b, d\rangle$. As a consequence, the column corresponding to $\langle b, d\rangle$ in the matrix of ∂_1 is the sum of the columns corresponding to the edges of the chain.

[13] As $\partial(k\langle z, w\rangle) = k\langle w\rangle - k\langle z\rangle$ this property holds for boundaries of single terms. By linearity of ∂ the same also holds for chains.

Assume $K_1, K_2, \dots, K_n$ are the components of K with $x_i \in K_i, \forall i$. We now combine the following information that allows us to describe $H_0(K;\mathbb{F})$:

1. Equality $\ker\partial_0 = C_0(K;\mathbb{F})$ means $\ker\partial_0 = Z_0(K;\mathbb{F})$ has a basis $\{\langle v\rangle\}_{v\in K^{(0)}}$.
2. For each edge $\langle x, y\rangle \in K$ we have $\partial\langle x, y\rangle = \langle y\rangle - \langle x\rangle$, meaning that $\langle x\rangle$ and $\langle y\rangle$ get identified in the homology group, i.e., $[\langle x\rangle] = [\langle y\rangle]$.
3. By Proposition 1.11 the equivalence classes of two vertices are identified in homology iff the vertices lie in the same components.
4. By 1. $\{[\langle v\rangle]\}_{v\in K^{(0)}}$ span $H_0(K;\mathbb{F})$ and by 2. and 3. so do $\{[\langle x_i\rangle]\}_{i=1}^{n}$.
5. The collection $\{[\langle x_i\rangle]\}_{i=1}^{n}$ is linearly independent, the proof of this claim being similar to the second part of the proof of Proposition 1.11.

As a result $\{[\langle x_i\rangle]\}_{i=1}^{n}$ is a basis of $H_0(X;\mathbb{F})$ and thus the dimension of $H_0(X;\mathbb{F})$ equals the number of components of K, i.e., $\mathfrak{b}_0 = n$. For example see Figure 10.

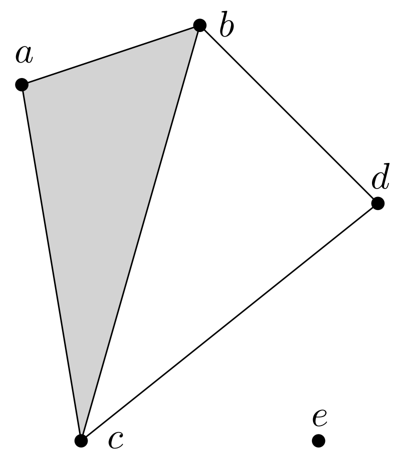

Figure 10: Abstract simplicial complex L. $H_0(L;\mathbb{F})$ is of dimension two (representing two components) with a basis being $[< a >] = [< b >] = [< c >] = [< d >]$ and $[< e >]$. $H_1(L;\mathbb{F})$ is of dimension one representing one hole, with a basis $[\langle c, d\rangle + \langle d, b\rangle + \langle b, c\rangle]$.

Homology of a graph

Let K be a simplicial complex which is a connected planar graph, and let $\mathbb{F}$ be any field. In this subsection we prove that <u>$\mathfrak{b}_1$ is the number of holes</u> K generates in the plane.

The homology group $H_1(K;\mathbb{F})$ is computed from the following piece of information:

$$C_2(K;\mathbb{F}) \xrightarrow{\partial_2} C_1(K;\mathbb{F}) \xrightarrow{\partial_1} C_0(K;\mathbb{F}).$$

As $0 = C_2(K;\mathbb{F})$ we have $H_1(K;\mathbb{F}) = \ker \partial_1$ so it suffices to determine the kernel of ∂_1.

1. Let $K' \leq K$ be a maximal tree with edges $e_1, e_2, \ldots, e_n$.

2. The collection $\partial e_1, \partial e_2, \ldots, \partial e_n$ is linearly independent by the following argument[14]. As $K' \simeq 0$ its first homology is trivial by Theorem 1.10 and as K' contains no triangles, $H_1(K';\mathbb{F}) = \ker \partial_1|_{C_1(K';\mathbb{F})}$. In particular, $\partial_1|_{C_1(K';\mathbb{F})}$ is injective. Its matrix contains $\partial e_1, \partial e_2, \ldots, \partial e_n$ as columns and injectivity implies the columns are linearly independent.

3. Let W denote the span of $\partial e_1, \partial e_2, \ldots, \partial e_n$.

4. Let $e_{n+1}, e_{n+1}, \ldots, e_m$ be the edges of K that are not contained in K', with each e_j being the edge from vertex x_j to vertex y_j.

5. Adding edges $e_{n+1}, e_{n+1}, \ldots, e_m$ to K' inductively, each addition of an edge increases the number of holes generated by the resulting graph by one.

6. In a parallel fashion, each addition of an edge increases the dimension of the kernel of the first boundary map by 1 as $\partial e_j \in W, \forall j \in \{n+1, n+1, \ldots, m\}$ by Proposition 1.11.

7. In the end of this process of adding edges we have generated $m-n$ holes and the dimension of $\ker \partial_1$ (and $\mathfrak{b}_1$) turns out to be $m-n$.

8. For each $j \in \{n+1, n+1, \ldots, m\}$ let c_j denote the (simplicial) path in K' from x_j to y_j represented as a 1-chain. The following form a basis of $H_1(K;\mathbb{F})$: $[e_j - c_j]$ for $j \in \{n+1, n+1, \ldots, m\}$.

   An example is displayed in Figure 12.

[14] For an alternative geometric argument see Figure 11.

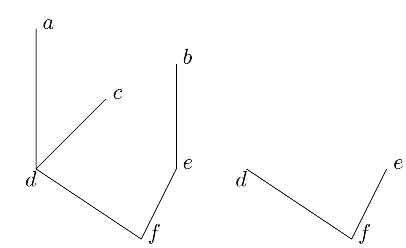

Figure 11: In this figure we demonstrate a geometric reason why the collection of the boundaries of all edges of a tree is linearly independent. Given a tree (on the left side of the figure) assume a linear combination of the boundaries of its edges is the zero vector. Since vertex a only appears in edge $\langle a, d\rangle$, the coefficient in front of that edge in the mentioned linear combination equals 0. The same argument holds for b and c and thus the mentioned linear combination only contains edges from the subtree on the right. Repeating the argument above, now for vertices d and e, we conclude that the mentioned linear combination is trivial and thus the claim holds. The same inductive argument works for any tree.

2 Computing homology

A systematic way to compute homology groups is through matrix reduction which allows us to obtain the rank[15] of a linear map. Before we provide the details on the rank computation, let us explain how to use it in order to compute the Betti numbers.

[15] Given a linear map of vector spaces, its rank is the dimension of its image.

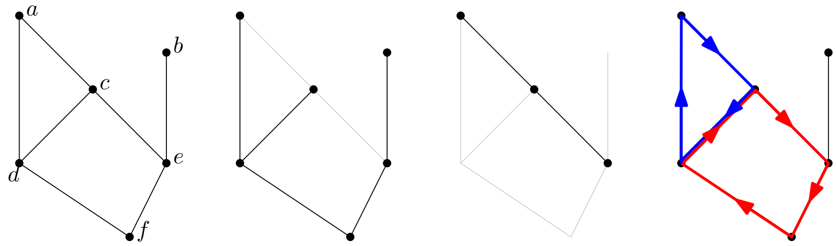

Figure 12: From left to right, the pictures represent a planar graph, a maximal tree, edges not contained in the chosen maximal tree and two cycles representing a basis of the first homology. Note that the graph induces two holes and thus $\mathfrak{b}_1 = 2$.

Let K be an abstract simplicial complex of dimension n and choose a field of coefficients $\mathbb{F}$. For each $p \in \{0, 1, \dots, n\}$ let n_p denote the number of simplices of dimension p in K.

Proposition 2.1. *Let $f\colon A \to B$ be a linear map of vector spaces. Then:*

1. $\dim A = \dim(\ker f) + \operatorname{rank} f$

2. $\dim(B/\operatorname{Im} f) = \dim B - \operatorname{rank} f$

Proposition 2.2. *1.* $\dim \ker \partial_p = n_p - \operatorname{rank} \partial_p$

2. $\mathfrak{b}_p = n_p - \operatorname{rank} \partial_p - \operatorname{rank} \partial_{p+1}$

Proof. Part 1. follows from 1. of Proposition 2.1 as $n_p = \dim C_p(K; \mathbb{F})$. Part 2. follows from 2. of Proposition 2.1 and 1. □

Part 1. of Proposition 2.1 is a standard statement of linear algebra. Part 2. was proved in the previous chapter.

Thus the Betti numbers can be expressed only using the number of simplices of a given dimension and the ranks of the corresponding boundary maps. Now let us turn out attention to rank computations. Given a boundary matrix, its rank[16] is easily obtained[17] from the row or column echelon form.

Echelon forms

In order to obtain a **row echelon form** of a matrix we can use the following operations[18]:

R1: exchange two rows;

R2: multiply[19] a row by a non-zero element of $\mathbb{F}$;

R3: add a multiple of one row to a different row.

C1: exchange two columns.

In the end we are aiming for the following transformation[20]

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix} \rightsquigarrow \overset{\overleftrightarrow{\;r\;}}{\begin{pmatrix} 1 & * & \cdots & \cdots & * \\ 0 & \ddots & & & \vdots \\ \vdots & \ddots & 1 & * \cdots & * \\ \vdots & & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}}$$

[16] And thus also the rank of the boundary map. Equivalent definitions of the rank of a matrix include: the maximal number of linearly independent columns; the maximal number of linearly independent rows.

[17] When using coefficients in $\mathbb{R}$ or $\mathbb{Q}$ the numerical procedure to obtain rank might in some cases result in certain instabilities. When using coefficients in $\mathbb{Z}_p$ however such issues do not arise, at least not for reasonably modest p.

[18] Of course, operations are considered in $\mathbb{F}$.

[19] Since $\mathbb{F}$ is a field, this means we can also divide a row by a non-zero element of $\mathbb{F}$.

[20] Symbols $*$ denote arbitrary elements of $\mathbb{F}$. The first r elements of the diagonal are declared to be 1 in our case. This is one version of the row echelon form and can always be achieved. However, there is a variant of a row echelon form in which these diagonal entries are non-zero, with the other $*$ entries still being arbitrary (possibly zero) elements of $\mathbb{F}$. Using this variant the rank is still obtained in the same way and as a benefit, the number of row operations required to reach it is typically smaller.

In the context of the Gaussian elimination a slightly different (and in general more common and slightly faster) row echelon form is usually computed without the use of operation C1.

The number of the non-trivial diagonal entries, r, equals to the rank of the matrix. In practice we will sometimes refrain from using $C1$ and only reduce to the classical row echelon form that is typically obtained through the Gaussian elimination.

In a similar way we can also compute the **column echelon form** using the corresponding column operations $C1, C2$, $C3$ and (possibly) $R1$.

Example 2.3. *Let us compute the homology of simplicial complex L from Figure 13. The boundary matrices are*

$$M_2 = \begin{array}{c} \\ \langle a,b\rangle \\ \langle b,c\rangle \\ \langle a,c\rangle \\ \langle b,d\rangle \\ \langle c,d\rangle \end{array} \begin{array}{c} \langle a,b,c\rangle \\ \left(\begin{array}{c} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{array}\right) \end{array}, \quad M_1 = \begin{array}{c} \\ \langle a\rangle \\ \langle b\rangle \\ \langle c\rangle \\ \langle d\rangle \\ \langle e\rangle \end{array} \begin{array}{c} \begin{array}{ccccc} \langle a,b\rangle & \langle b,c\rangle & \langle a,c\rangle & \langle b,d\rangle & \langle c,d\rangle \end{array} \\ \left(\begin{array}{c|c|c|c|c} -1 & & -1 & & \\ 1 & -1 & & -1 & \\ \hline & 1 & 1 & & -1 \\ \hline & & & 1 & 1 \\ & & & & \end{array}\right) \end{array}.$$

Performing only row operations we obtain[21]

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \textit{and} \quad \left(\begin{array}{c|c|c|c|c} 1 & & 1 & & \\ & 1 & 1 & 1 & \\ \hline & & & 1 & 1 \\ \hline & & & & \\ & & & & \end{array}\right)$$

These are the classical row echelon forms typically obtained through the Gaussian reduction[22] *and the rank of such a matrix is the number of pivots*[23]*. The corresponding ranks of the matrices are* 1 *and* 3*. We thus have* $\operatorname{rank}\partial_2 = 1, \operatorname{rank}\partial_1 = 3, n_2 = 1, n_1 = 5, n_0 = 5$ *and we conclude:*

- $\mathfrak{b}_2 = n_2 - \operatorname{rank}\partial_2 = 0$*, the complex encloses no "void".*
- $\mathfrak{b}_1 = n_1 - \operatorname{rank}\partial_1 - \operatorname{rank}\partial_1 = 1$*, which is the number of holes.*
- $\mathfrak{b}_0 = n_0 - \operatorname{rank}\partial_1 = 2$*, which is the number of components.*

Smith normal form and representatives

While the computation of the echelon forms suffices to compute the Betti numbers, we are often interested in the representing cycles[24] of homology groups as well. To that end we employ a different canonical form of a matrix: the **Smith normal form**. It is obtained from the row echelon form by eliminating the $*$ entries to zero using the mentioned

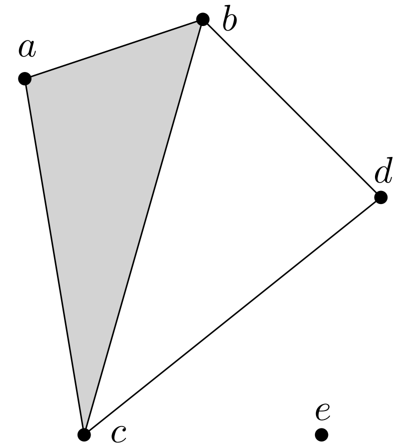

Figure 13: Simplicial complex L.

[21] The reduced form in this case coincides for all fields $\mathbb{F}$. Later we will see, for example with the Klein bottle, that the reduced forms and ranks in general depend on $\mathbb{F}$.

[22] In order to obtain pivots only on the diagonal, as the row echelon form as we defined it requires, we would need to exchange columns 3 and 4.

[23] Equivalently, the number of non-zero rows.

☞ Row and column operations geometrically amount to changes in the bases of the domain and target vector spaces. These changes can be encoded in transformation matrices and in fact, most special forms or reductions of matrices are often expressed in terms of matrix factorizations. For our illustrative purposes though we will stick with the annotations.

[24] There are also other ways to compute the representing cycles although, at the end of the day, most of them use a similar amount of linear algebra. A high-level approach would be the following. First compute the basis of $\operatorname{Im}\partial_{p+1}$, which is the column space of the corresponding boundary matrix. Then complete it to the basis of $\ker\partial_p$. The vectors forming the completion represent the basis of p-homology. As mentioned, there are many ways to practically formalize these steps, including the presented one through the Smith normal form.

row and column operations $R1, R2, R3, C1, C2, C3$.

$$
\begin{pmatrix}
a_{1,1} & \cdots & a_{1,n} \\
\vdots & \ddots & \vdots \\
a_{m,1} & \cdots & a_{m,n}
\end{pmatrix}
\rightsquigarrow
\begin{array}{c}
\overset{r}{\longleftrightarrow} \\
\begin{pmatrix}
1 & 0 & \cdots & & 0 \\
0 & \ddots & & & \vdots \\
\vdots & & 1 & 0 \cdots & 0 \\
 & & 0 & \cdots & 0 \\
\vdots & & & & \vdots \\
0 & \cdots & & & 0
\end{pmatrix}
\end{array}
$$

In order to obtain the representing cycles though, we need to use the annotated rows and columns:

- The annotations of columns from index $r+1$ on form the basis of the kernel.
- The boundaries of annotations of columns of index up to r on form the basis of the image.

$$
\begin{array}{c}
\qquad\qquad \overset{\ker}{\longleftrightarrow} \\
\begin{pmatrix}
1 & 0 & \cdots & & 0 \\
0 & \ddots & & & \vdots \\
\vdots & & 1 & 0 \cdots & 0 \\
 & & 0 & \cdots & 0 \\
\vdots & & & & \vdots \\
0 & \cdots & & & 0
\end{pmatrix}
\end{array}
$$

Example 2.4. *Let us compute the representatives of the homology groups of simplicial complex L from Figure 14. The annotated boundary matrices are*

$$
M_2 = \begin{array}{c|c|}
 & \langle a,b,c\rangle \\
\langle a,b\rangle & 1 \\
\langle b,c\rangle & 1 \\
\langle a,c\rangle & -1 \\
\langle b,d\rangle & 0 \\
\langle c,d\rangle & 0
\end{array},
\quad
M_1 = \begin{array}{c|c|c|c|c|c|}
 & \langle a,b\rangle & \langle b,c\rangle & \langle a,c\rangle & \langle b,d\rangle & \langle c,d\rangle \\
\langle a\rangle & -1 & & -1 & & \\
\langle b\rangle & 1 & -1 & & -1 & \\
\hline
\langle c\rangle & & 1 & 1 & & -1 \\
\hline
\langle d\rangle & & & & 1 & 1 \\
\langle e\rangle & & & & &
\end{array}
$$

with the annotated row echelon forms being[25]

$$
\begin{array}{c}
\langle a,b,c\rangle \\
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\end{array}
\quad \text{and} \quad
\begin{array}{c|c|c|c|c}
\langle a,b\rangle & \langle b,c\rangle & \langle b,d\rangle & \langle a,c\rangle & \langle c,d\rangle \\
1 & & & 1 & \\
 & 1 & 1 & 1 & \\
\hline
 & & 1 & & 1 \\
\hline
 & & & &
\end{array}.
$$

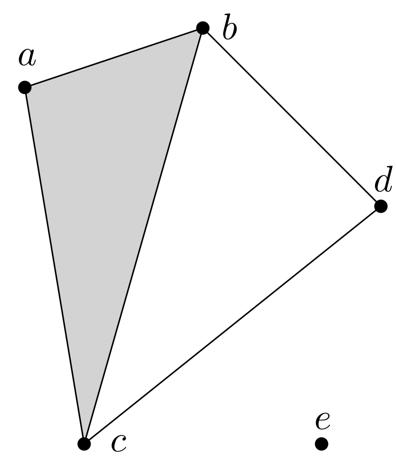

Figure 14: Abstract simplicial complex L.

[25] Only the column annotations will be displayed as the row annotations are not required.

The first of these two matrices is already in the Smith normal form. The Smith normal form of the second matrix is:

$$
\begin{array}{ccccc}
\scriptstyle\langle a,b\rangle & \scriptstyle\langle b,c\rangle & \scriptstyle\langle b,d\rangle-\langle b,c\rangle & \scriptstyle\langle a,c\rangle-\langle b,c\rangle-\langle a,b\rangle & \scriptstyle\langle c,d\rangle-\langle b,d\rangle+\langle b,c\rangle \\
\end{array}
$$
$$
\left(\begin{array}{c|c|c|c|c}
1 & & & & \\
 & 1 & & & \\
\hline
 & & 1 & & \\
\hline
 & & & & \\
\end{array}\right).
$$

We now construct the homology representatives by dimension:

Dimension 0*:*

1. $\ker \partial_0$ *has a basis* $\langle a\rangle, \langle b\rangle, \langle c\rangle, \langle d\rangle, \langle e\rangle$.
2. $\operatorname{Im} \partial_1$ *has a basis formed by the images of the first three annotated columns of the Smith normal form, i.e.,* $\langle a,b\rangle, \langle b,c\rangle$*, and* $\langle b,d\rangle - \langle b,c\rangle$*. The basis obtained in this way is*
$$\langle b\rangle - \langle a\rangle, \langle c\rangle - \langle b\rangle, \text{ and } \langle d\rangle - \langle b\rangle + \langle c\rangle - \langle b\rangle.$$
3. *We may complete the basis from 2. to the basis of* $\ker \partial_0$ *by, for example, adding* $\langle a\rangle$ *and* $\langle e\rangle$ *and thus* $\langle a\rangle$ *and* $\langle e\rangle$ *represent the two* 0*-holes*[26] *spanning* $H_0(L;\mathbb{F})$.

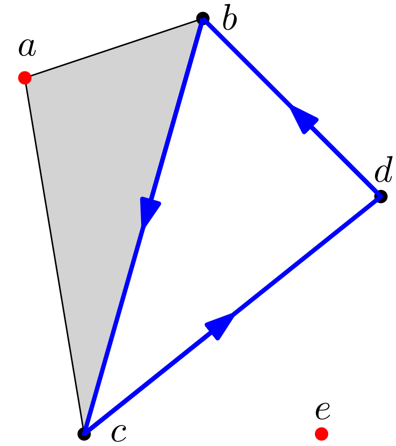

Figure 15: Obtained representatives of bases of the homology groups of L. Representatives $\langle a\rangle$ and $\langle e\rangle$ in red spanning $H_0(L;\mathbb{F})$, and representative $\langle c,d\rangle - \langle b,d\rangle + \langle b,c\rangle$ in blue spanning $H_0(L;\mathbb{F})$.

[26] I.e., components.

Dimension 1*:*

1. $\ker \partial_1$ *has a basis* $\langle a,c\rangle - \langle b,c\rangle - \langle a,b\rangle$ *and* $\langle c,d\rangle - \langle b,d\rangle + \langle b,c\rangle$.
2. $\operatorname{Im} \partial_1$ *has a basis formed by the images (boundaries) of the first annotate column of the Smith normal form, i.e.,* $\langle a,b,c\rangle$*. The basis obtained in this way is*
$$\langle a,b\rangle + \langle b,c\rangle - \langle a,c\rangle.$$
3. *We may complete*[27] *the basis from 2. to the basis of* $\ker \partial_0$ *by, for example, adding* $\langle c,d\rangle - \langle b,d\rangle + \langle b,c\rangle$ *and thus* $\langle c,d\rangle - \langle b,d\rangle + \langle b,c\rangle$ *represents a* 1*-holes spanning* $H_1(L;\mathbb{F})$.

[27] The fact that the basis element from 2. is a member from the basis of 1. helps us to see this completion immediately. However, such a situation is an exception and a completion of basis typically involves some work with linear algebra.

Incremental expansion and elementary collapse

We conclude the section by analysing how a minimal change to a simplicial complex, an addition of one or two simplices, affects the homology.

We first discuss the **incremental expansion**, or how an addition[28] of a simplex to a simplicial complex changes the homology. Let K be a simplicial complex and let $\sigma^{(n)} \notin K$ be an n-simplex on vertices of K such that $K \cup \{\sigma\}$ is[29] also a simplicial complex. The addition of σ to K has the following effect to the homology computation scheme:

[28] Or a removal, which can be analyzed in a similar fashion.

[29] In particular, all faces of σ should be present in K.

1. The number of n-simplices increases[30] by 1.

2. If chain $\partial\alpha$ is already contained in $\ker \partial_n$, then the addition of σ to the boundary matrix of ∂_n adds a column, which is linearly dependent on other columns and in effect, the dimension of the kernel is increased by 1.

3. If chain $\partial\alpha$ is not in $\ker \partial_n$, then the addition of σ to the boundary matrix of ∂_n adds a column, which is linearly independent on other columns and in effect, the rank of the matrix is increased by 1.

As a result (see Figure 16), an incremental expansion either increases $\mathfrak{b}_n$ by 1 (case 2.), or decreases $\mathfrak{b}_{n-1}$ by 1 (case 3.).

[30] This means that either the dimension of the kernel of ∂_n or the rank of ∂_n increases by 1.

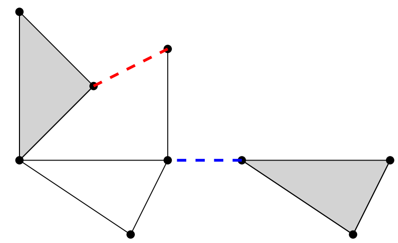

Figure 16: A demonstration of incremental expansion. Adding an edge to a simplicial complex may either reduce $\mathfrak{b}_0$ (the number of components) by 1 (blue case) or increase $\mathfrak{b}_1$ (the number of holes) by 1 (red case).

We next discuss the **elementary collapse**. We have already mentioned it in the chapter on simplicial complexes. Let K be a simplicial complex, $\tau^{(k-1)} \subset \sigma^{(k)} \in K$, and assume σ is the only coface of τ. A removal $K \to K \setminus \{\tau, \sigma\}$ is called an elementary collapse. It is a modification that does not change the homotopy type, and hence the homology is preserved.

Let us see how an elementary collapse effects the computation of homology.

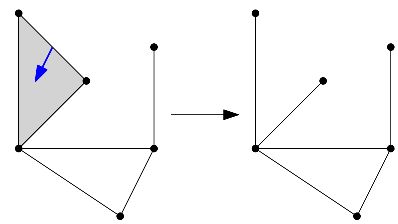

Figure 17: An elementary collapse.

- The boundary of σ is not a linear combination of boundaries of other k-simplices as σ is the only[31] coface of τ. Hence removing σ decreases $\operatorname{rank} \partial_k$ by 1.

- The boundary of τ is a linear combination of boundaries of other $(k-1)$-simplices by the following argument. Simplex τ is contained[32] in the chain $\partial\sigma$. Since the boundary of this chain equals zero[33], we can express $\partial\tau$ as a sum of boundaries of other facets of σ with the appropriate coefficients ± 1. Hence τ is a linear combination of boundaries of other $(k-1)$-simplices and thus removing it decreases $\ker \partial_{k-1}$ by 1.

[31] Meaning that $\partial\sigma$ is the only boundary of a k-simplex containing a term with τ.

[32] ...with coefficient $+1$ or -1.

[33] $\partial^2\sigma = 0$

In total, the dimensions of the homology groups do not[34] change.

[34] Recall that the only homology group that may potentially change is H_{k-1}. It is defined as $\ker \partial_{k-1} / \operatorname{Im} \partial_k$ and since the dimension of both $\ker \partial_{k-1}$ and $\operatorname{Im} \partial_k$ decreases by one, the dimension of the quotient is preserved.

3 Examples of homology

In this section we present some further aspects of homology that should aid our understanding of the concept.

Disjoint unions

Two abstract simplicial complexes are said to be **disjoint** if their collections are disjoint[35]. Two geometric simplicial complexes are said to be **disjoint** if their bodies are disjoint. The union of disjoint

[35] I.e., if there is no intersection between the sets of vertices. Formally speaking, if such an intersection existed it would mean that we are treating both collection of vertices as subsets of some larger set.

simplicial complexes K, L is called a **disjoint union** and is denoted by $K \coprod L$.

Given two disjoint simplicial complexes K, L, the homology of their disjoint union is the cartesian product[36] of the individual homologies: $H_i(K \coprod L; G) \cong H_i(K; G) \times H_i(K; G)$. Computationally we can see this by observing that the boundary map ∂ has block-diagonal matrices: boundaries of chains from K lie in K and the same holds for L. Since each simplicial complex is the disjoint union of its components, the technical computations and treatments of homology are typically restricted to connected simplicial complexes.

[36] In our setting, the term "direct sum" could also be used.

Example 3.1. *Given a planar graph K and any field $\mathbb{F}$:*

- $\mathfrak{b}_0$ *is the number of components of K.*
- $\mathfrak{b}_1$ *is the number of holes of K induces in the plane.*

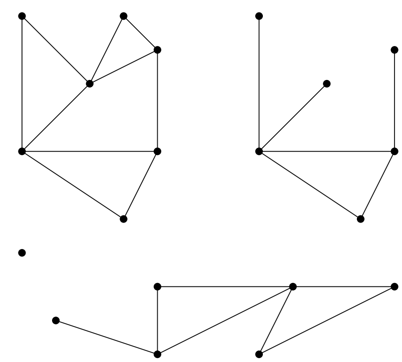

Figure 18: A planar graph with four components: $\mathfrak{b}_0 = 4, \mathfrak{b}_1 = 7, \chi = -3$.

Euler characteristic

Suppose K is a simplicial complex and let n_i denote the number of i-simplices in K. Recall that the Euler characteristic $\chi(K) \in \mathbb{Z}$ is defined as $\chi(K) = n_0 - n_1 + n_2 - n_3 + \dots$.

This invariant has an interesting interpretation in terms of homology.

Proposition 3.2. $\chi(K) = \mathfrak{b}_0 - \mathfrak{b}_1 + \mathfrak{b}_2 - \mathfrak{b}_3 + \dots$.

Proof. By 2. of Proposition 2.2 we have $\mathfrak{b}_p = n_p - \operatorname{rank} \partial_p - \operatorname{rank} \partial_{p+1}$. Substituting these equality into $\mathfrak{b}_0 - \mathfrak{b}_1 + \mathfrak{b}_2 - \mathfrak{b}_3 + \dots$ we obtain χ. $\square$

Example 3.3. *Given a planar graph K and any field $\mathbb{F}$, $\chi(K)$ equals the number of components subtracted by the number of holes K generates in the plane.*

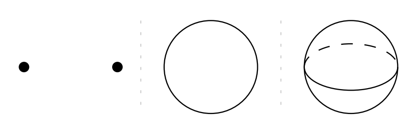

Figure 19: S^0 demonstrates non-trivial H_0, S^1 represents a one-dimensional hole, and S^2 encloses a two-dimensional hole.

Spheres

Holes as measured by homology are represented by cycles and the fundamental examples of holes are provided[37] by spheres. In this subsection we prove that given a triangulation of an n-sphere for $n \geq 1$, the consistently oriented collection of n-simplices represents an n-hole. In fact, this is the only hole a sphere has. A convenient triangulation of S^n we will be using will be the one[38] consisting of all faces of an $(n+1)$-simplex.

[37] The homology of a metric space is, for our purposes, the homology of any triangulation of that space.

[38] To be precise: take an $(n+1)$-simplex, add all of its faces to obtain a simplicial complex called the **full simplex on** $n+2$ **points** (sometimes also called the **full** $(n+1)$**-simplex**), and then remove the $(n+1)$-simplex to obtain a trianagulation of S^n.

Proposition 3.4. *For each* $\mathbb{F}$ *and* $n \in \{1, 2, \ldots\}$ *we have:*

- $H_0(S^n; \mathbb{F}) \cong H_n(S^n; \mathbb{F}) \cong \mathbb{F}$;
- $H_i(S^n; \mathbb{F}) = 0, \forall i \notin \{0, n\}$.

Proof. The full simplicial complex on $n+2$ points is contractible hence all its homology groups are trivial except for H_0, which is of rank 1. Removing the only $(n+1)$-simplex reduces $\mathsf{rank}\, \partial_{n+1}$ by one and hence increases $\mathfrak{b}_n$ by 1, as was explained in the context of incremental expansions and removals. □

☞ Fun observation: The full simplex on n points is contractible hence its Euler characteristic equals 1. On the other hand, computing its Euler characteristic by definition we get n points $-\binom{n}{2}$ edges $+ \binom{n}{3}$ triangles $\ldots (-1)^n \cdot 1$ n-simplex. Summing it up we get:

$$\binom{n}{1} - \binom{n}{2} + \binom{n}{3} - \ldots (-1)^n \binom{n}{n} = 1,$$

which is a special case of the binomial formula.

Given that S^0 is a collection of two points it is easy to see[39] that the only non-trivial homology group of S^0 is $H_0(S^0; \mathbb{F}) \cong \mathbb{F}^2$.

[39] ..by a direct computation or by the argument of Proposition 3.4.

Surfaces

A beautiful demonstration of the two-dimensional homology is provided by surfaces.

Proposition 3.5. *Let K be a triangulation of a closed (i.e., without boundary) connected orientable surface. For each group* $\mathbb{F}$ *we have* $H_2(K; \mathbb{F}) \cong \mathbb{F}$.

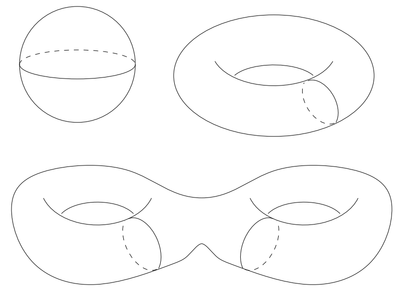

Figure 20: Examples of closed connected surfaces: they all enclose one 2-dimensional hole in the form of a "cave", which is manifested in the fact that $\mathfrak{b}_2 = 1$.

Proof. Recall that K being orientable means there exists a consistent choice of orientations on all triangles of K. Let us fix such an orientation on them.

1. The structure of a surface implies that each edge of K is a face of at most two triangles.
2. The structure of a *closed* surface implies that each edge of K is a face of precisely two triangles.
3. Consistency of orientations on triangles implies that whenever two triangles intersect in an edge, the induced orientations on the edge are the opposite.

⚠ The statement of Proposition 3.5 does not hold for connected surfaces with boundary. If there was a non-trivial 2-cycle in such a case, the same argument as in the proof of the proposition would imply that the cycle would be the oriented sum of all triangles (possibly multiplied by a single non-trivial factor $\lambda \in \mathbb{F}$). Since a presence of a boundary of a manifold implies the existence of an edge, which is a face of precisely one triangle, such a triangle (multiplied by λ) would thus appear in the boundary of the cycle, a contradiction.

The second homology of a connected manifold with a boundary is thus always trivial.

Let us define chain α as the sum of all oriented triangles. By 1.-3. above each edge appears in $\partial\alpha$ twice, once with each orientation (see Figure 21), and thus $\partial\alpha = 0$, meaning that α is a chain. As the image of ∂_3 is trivial, α represents a non-trivial homology class.

On the other hand, whenever a 2-cycle β contains a term[40] $+\sigma$ where σ an oriented triangle, observations 2. and 3. imply that all oriented simplices sharing an edge with σ also appear in α with coefficient 1. Inductively expanding this conclusion to further neighbors we

[40] If σ appear in the term $\lambda\sigma$ for some non-zero $\lambda \in \mathbb{F}$, we repeat the same argument for the chain divided by λ.

reach all triangles as K is connected and thus deduce that $\beta = \alpha$. The proposition is thus proved. $\square$

Homology class $[\alpha]$ generating $H_2(K;\mathbb{F})$ as defined[41] in the proof is called the **fundamental class** of a surface. In the same way we can prove that if K is a closed connected orientable manifold of dimension n, then $H_n(K;\mathbb{F}) \cong \mathbb{F}$ with the generator, which is again called the fundamental class, being the sum of all consistently oriented n-simplices of K.

[41] Formally speaking, there are two fundamental classes, one for each orientation of triangles...except when $\mathbb{F} = \mathbb{Z}_2$. If $\mathbb{F} = \mathbb{Z}_2$ there is only one non-trivial homology class which is its own converse.

The case of non-orientable surfaces is the first presented situation in which the choice of coefficients matters.

Proposition 3.6. *Let K be a triangulation of a closed connected non-orientable surface. Then $H_2(K;\mathbb{Z}_2) \cong \mathbb{Z}_2$ and $H_2(K;\mathbb{F}) \cong 0$ for each $\mathbb{F} \ncong \mathbb{Z}_2$.*

☞ Proposition 3.6 also generalizes to the n-dimensional homology of closed connected non-orientable n-manifolds.

Proof. As in the proof of Proposition 3.5, the fact that K is a surface means that if a 2-cycle α contains a term $+\sigma$ for some oriented triangle σ, it also contains a term $+\sigma'$ for each oriented triangle σ' sharing[42] an edge with σ. Again, as K is connected, this means that α is the sum of all oriented triangles. However, as K is non-orientable, there is no consistent orientation on triangles and thus[43] some edges appear with coefficient 2 in the boundary, see Figure 21. Thus if[44] $0 \neq 2$ the boundary is non-trivial and the assumed 2-cycle does not have the empty boundary, a contradiction. Hence the only cycle is the trivial cycle.

However, if $\mathbb{F} = \mathbb{Z}_2$, the obtained boundary equals zero and thus α is the only non-trivial cycle. As a result, $H_2(K;\mathbb{Z}_2) \cong \mathbb{Z}_2$. $\square$

[42] Sharing in the sense of consistent orientation, meaning that the induced orientation on the shared edge are the opposite.

[43] As each edge appears twice in the boundary of such a chain and not all such appearances may cancel each other out by the non-orientability.

[44] Equivalently, if $\mathbb{F} \neq \mathbb{Z}_2$.

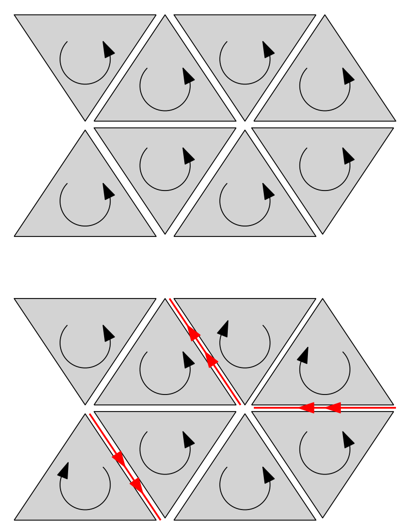

Figure 21: Top: The boundary of a chain consisting of all consistently oriented triangles of a surface without boundary is zero, as the induced orientations on edges cancel out. Bottom: The boundary of a chain consisting of a not-consistently oriented collection of all triangles of a surface without boundary contains each edge between two non-consistently oriented triangles twice.

We may summarize these two propositions and the corresponding comments as follows:

- A connected surface K is closed iff $H_2(K;\mathbb{Z}_2) \neq 0$.
- Given any field $\mathbb{F} \neq \mathbb{Z}_2$, a closed connected surface is orientable if $H_2(K;\mathbb{F}) \cong \mathbb{F}$.

Impact of coefficients: the Klein bottle

Another example where the choice of coefficients makes a difference in homology computations is the Klein bottle, which will be denoted by $\mathcal{K}$ in this subsection. It is depicted in Figure 22. Its triangulation is given by the black portion in Figure 24. We already know that $\mathfrak{b}_0 = 1$ as $\mathcal{K}$ is connected. However, the second Betti number of this closed surface depends on the coefficients due to the non-orientability:

- $\mathfrak{b}_2(\mathcal{K};\mathbb{Z}_2) = 1$.
- For $\mathbb{F} \ncong \mathbb{Z}_2$, $\mathfrak{b}_2(\mathcal{K};\mathbb{F}) = 0$.

From this information and the expression of the Euler characteristic as the alternative sum of Betti numbers we conclude:

- $\mathfrak{b}_1(\mathcal{K};\mathbb{Z}_2) = 2$, i.e., $H_2(\mathcal{K};\mathbb{Z}_2) \cong \mathbb{Z}_2^2$.
- For $\mathbb{F} \ncong \mathbb{Z}_2$, $\mathfrak{b}_1(\mathcal{K};\mathbb{F}) = 1$.

These Betti numbers can also be computed through the matrix reduction. Instead of computing them, we will rather demonstrate the geometric reason for the difference in $\mathfrak{b}_1$ depending on the coefficients. The explanation will be based on Figure 24. On each of the five parts of the figure a triangulation of $\mathcal{K}$ is provided by the black/grey portion. The black arrows indicate the direction in which the identifications are performed. A single red horizontal directed line represents a cycle α generating the extra dimension of $H_2(\mathcal{K};\mathbb{Z}_2)$. It is also depicted in Figure 22. It turns out that $[\alpha]$ is homologically non-zero iff the used coefficients are $\mathbb{Z}_2$. In order to prove this statement we first present a claim.

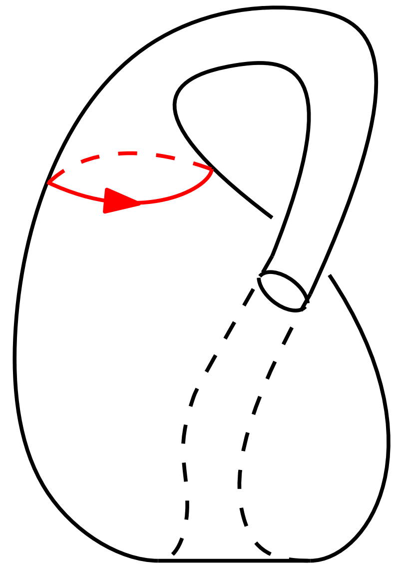

Figure 22: The Klein bottle.

We claim that $2[\alpha]$ is the trivial homology class. In order to prove the claim the leftmost part of Figure 24 has two copies of α drawn slightly apart from each other for better distinguishability. The corresponding homology class does not change if we move[45] each of the copies of α separately. So let's move them as on the Figure:

- move the upper copy slightly higher;
- move the lower copy to the bottom of the side. Due to the reversed orientation, the chain then appears on the top of the square with (in the plane seemingly) reversed orientation. Moving this representative lower to the first copy of α we see, that the copies cancel each other out: they consist of the same edges with converse orientations.

[45] "Moving" in this setting can be thought of as a homotopic change. Formally speaking, a moved chain represents the same homology class if the difference between the original and the new chain is in the boundary group, see Figure 23.

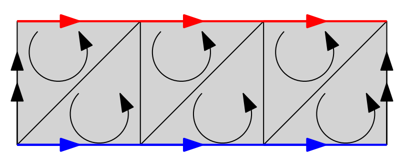

Figure 23: Excerpt from the transformation in Figure 24. The blue and the red chain represent the same homology class because their difference (blue − red) is the boundary of the 2-chain consisting of the strip of depicted oriented triangles.

As a result, the claim holds, i.e, $2[\alpha]$ is the trivial homology class. Depending on the coefficients of our computation this has the following ramifications:

- if $\mathbb{F} \ncong \mathbb{Z}_2$ then we can divide equation $2[\alpha] = 0$ by 2 and obtain that $[\alpha] = 0 \in H_1(\mathcal{K};\mathbb{F})$.
- if $\mathbb{F} \cong \mathbb{Z}_2$ then we can't divide equation $2[\alpha] = 0$ by 2 as $2 = 0$. It turns out that $[\alpha] \neq 0 \in H_1(\mathcal{K};\mathbb{Z}_2)$ and thus α provides an extra dimension to $H_1(\mathcal{K};\mathbb{Z}_2)$.

For an alternative argument proving the claim see Figure 25.

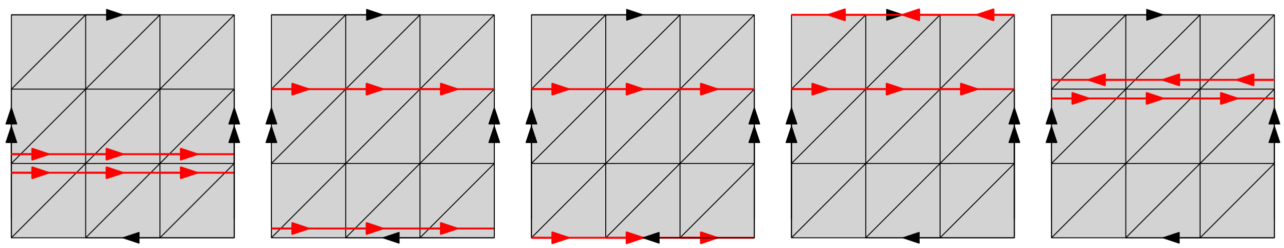

Figure 24: The Klein bottle.

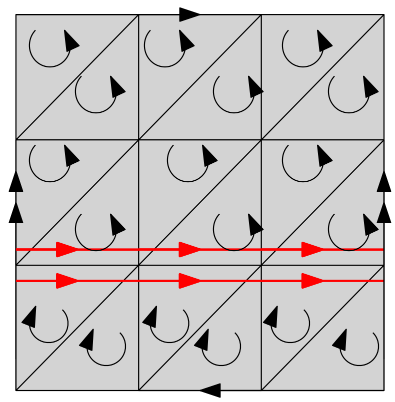

Figure 25: Another proof of the fact that $2[\alpha]$ is homologically trivial within the Klein bottle. The chain $2[\alpha]$ is depicted in red and is the boundary of the 2-chain consisting of all depicted oriented triangles.

Alexander duality

Homology is defined for any abstract simplicial complex. However, if there is an underlying geometric simplicial complex K embedded in a sphere or an Euclidean space, there is a connection between the homology of K and that of its complement. The relationship is formally known as the Alexander duality.

Before we state the duality we should elucidate a few technical details of the complement construction. Let $K \subset \mathbb{R}^2$ be a geometric simplicial complex. In particular, K consists[46] of finitely many simplices. The complement of K, denoted by $K^{\mathsf{C}} = \mathbb{R}^2 \setminus K$, is unfortunately not homeomorphic to a (finite[47]) simplicial complex. As a proof of this claim observe that K is a closed[48] subset of the plane, while K^{C} is usually[49] not. However, K^{C} is homotopic to a finite simplicial complex. For example see Figure 26. At his point we defer from specifying details of triangulation of K^{C} or its homotopy type and rather conclude with the geometrically declaration: K^{C} is homotopy equivalent to a finite simplicial complex K' and so whenever we will be talking about the homology of K^{C}, we will formally be thinking of the homology of K'. The same discussion applies if K is a geometric simplicial complex in any Euclidean space of a sphere.

The Alexander duality provides a connection between the homologies of K and its complement.

Theorem 3.7 (Alexander duality)**.** *Let $n \in \mathbb{N}$ and suppose $K \subset S^n$ is a geometric simplicial complex. Then for any coefficients $\mathbb{F}$ we have:*

1. $\mathfrak{b}_0(K;\mathbb{F}) - 1 = \mathfrak{b}_{n-1}(K^{\mathsf{C}};\mathbb{F})$.
2. $\mathfrak{b}_{n-1}(K;\mathbb{F}) = \mathfrak{b}_0(K^{\mathsf{C}};\mathbb{F}) - 1$.
3. $\mathfrak{b}_q(K;\mathbb{F}) = \mathfrak{b}_{n-q-1}(K^{\mathsf{C}};\mathbb{F})$ *for all* $q \in \{1, 2, \ldots, n-1\}$.

From the Alexander duality we may draw a similar conclusion for complexes in Euclidean spaces by taking into account that removing a

46 Formally, the body of K is the union of finitely many simplices.

47 Recall that all simplicial complexes considered here are finite. Within the context of infinite simplicial complexes though, the complement can be triangulated and the treatment of complements presented here is immaterial.

48 In particular, this means that the limit of each converging sequence in K lies in K.

49 Except if K is empty.

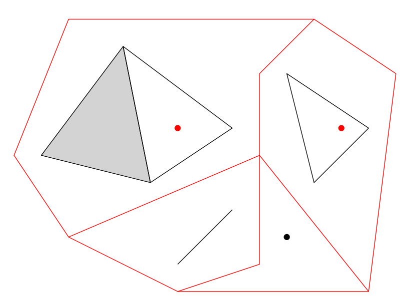

Figure 26: Simplicial complex K in the plane in black, and a simplicial complex K' homotopy equivalent to its complement in red. Note the number of holes of K is one less than the number of components of K', i.e., $\mathfrak{b}_1(K) = \mathfrak{b}_0(K') - 1$. Also, number of holes of K' equals the number of components of K, i.e., $\mathfrak{b}_1(K') = \mathfrak{b}_0(K)$.

point[50] from S^n results in a space homeomorphic to $\mathbb{R}^n$.

[50] A removal of a point from $K^C \subset S^n$ increases $\mathfrak{b}_{n-1}$ by one.

Corollary 3.8. *Let $n \in \mathbb{N}$ and suppose $K \subset \mathbb{R}^n$ is a geometric simplicial complex. Then for any coefficients $\mathbb{F}$ we have:*

1. $\mathfrak{b}_0(K;\mathbb{F}) = \mathfrak{b}_{n-1}(K^C;\mathbb{F})$.
2. $\mathfrak{b}_{n-1}(K;\mathbb{F}) = \mathfrak{b}_0(K^C;\mathbb{F}) - 1$.
3. $\mathfrak{b}_q(K;\mathbb{F}) = \mathfrak{b}_{n-q-1}(K^C;\mathbb{F})$ *for all* $q \in \{1,2,\dots,n-1\}$.

☞ Proofs of Alexander duality are quite technical and typically involve cohomology.

Alexander duality is handy when computing homology groups of simplicial complexes in Euclidean spaces or spheres. For example, instead of computing the one-dimensional homology of a planar simplicial complex, we can[51] compute the number of components of its complement, which is typically much faster.

[51] Provided there is an easy description of a complement. Such examples would include bitmap images.

4 Concluding remarks

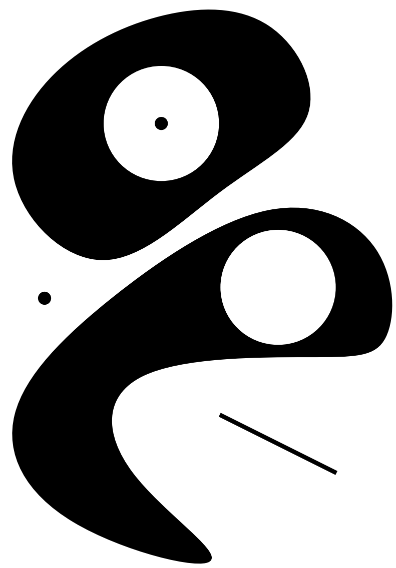

Figure 27: A demonstration of Alexander duality: given a bounded subset X of the plane, each component of X corresponds to a hole in X^C, and each hole in X corresponds to bounded component of X^C.

Recap (highlights) of this chapter

- Cycles, boundaries, homology
- Detecting components and holes with homology
- Computing homology through matrix reduction
- Euler characteristic
- Alexander duality

Background and applications

Homology is one of the focal invariants in topology and geometry. Homological conditions and constructions can be found throughout mathematics. We will present one of them in the appendix (Cubical homology). The version presented here is usually called "simplicial homology" as it arises from the structure of a simplicial complex. For non-triangulated spaces a version called "singular homology" can be defined. In general though, any reasonable boundary map ∂ satisfying ∂^2 induces its own homology structure. Examples[52] include cubical homology (see appendix) and cohomology.

[52] Another example is the De Rham cohomology and exterior derivative. While the theory itself is quite involved, a snapshot of the fact that $\partial^2 = 0$ can be observed in low dimensions via specific derivatives: gradient, divergence, and curl, are specific boundary maps as the composition of a consecutive pair amongst them equals zero.

Amenability to algorithmic computations through matrix reductions and, as we will see later, Discrete Morse Theory makes homology an obvious tool with which we could determine topological properties of data. In practice though the usual homology is often superseded by persistent homology, which is a richer, parameterized version of homology described in later chapters.

Appendix: Homology with coefficients in Abelian groups

Classical introductions of homology typically consider coefficients from an Abelian group rather than a field. By far the most popular choice among non-fields is the group of integers $\mathbb{Z}$. In this subsection we review the construction and properties of homology using coefficients[53] in a group $\mathbb{Z}$.

[53] The presented treatment would be practically identical for any Abelian group as the coefficient group.

Let K be an abstract simplicial complex of dimension n. For each $q \in \{0, 1, \ldots, n\}$ let n_q denote the number of simplices of dimension q in K.

The definition of homology in this case remains the same with the only difference being that the structure of algebraic invariants is that of Abelian groups, and the boundary operator ∂ is a homomorphism:

1. A q-chain is a formal sum $\sum_{i=1}^{n_q} a_i \sigma_i^q$ where $a_i \in \mathbb{Z}$ and σ_i^q is an oriented simplex of dimension q in K.

2. The chain group $C_q(K; \mathbb{Z}) \cong \mathbb{Z}^{n_q}$ is the group of all q-chains. Its generators are oriented q-simplices of K.

3. For each $p \in \mathbb{N}$ the boundary map
$$\partial_p \colon C_p(K; \mathbb{Z}) \to C_{p-1}(K; \mathbb{Z})$$
is the homomorphism defined by
$$\partial_p \langle v_0, v_1, \ldots, v_p \rangle = \sum_{i=0}^{p} (-1)^i \langle v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_p \rangle.$$
As before, $\partial^2 = 0$. Additionally define $\partial_0 = 0$.

4. The collection of chain groups bound together by the boundary homomorphisms is called the chain complex:
$$\cdots \xrightarrow{\partial} C_n(K; \mathbb{Z}) \xrightarrow{\partial} C_{n-1}(K; \mathbb{Z}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_1(K; \mathbb{Z}) \xrightarrow{\partial} C_0(K; \mathbb{Z}) \xrightarrow{\partial} 0$$

5. For each $q \in \{0, 1, \ldots\}$. We define groups:
   - q-cycles as $Z_q(K; \mathbb{Z}) = \ker \partial_q \leq C_q(K; \mathbb{Z})$.
   - q-boundaries as $B_q(K; \mathbb{Z}) = \operatorname{Im} \partial_{q+1} \leq Z_q(K; \mathbb{Z}) \leq C_q(K; \mathbb{Z})$.
   - q-homology group as the quotient $H_q(K; \mathbb{Z}) = Z_q(K; \mathbb{Z}) / B_q(K; \mathbb{Z})$.

Proposition 4.1. *Suppose G, H are Abelian groups, a map $f \colon G \to H$ is a homomorphism, and $G' \leq G$. Then:*

1. $\operatorname{Im}(f) \cong G / \ker(f)$.

2. $\operatorname{rank}(G/G') = \operatorname{rank}(G) - \operatorname{rank}(G')$.

Up to this point the introduction has been analogous to the one where coefficients form a field. However, as $H_q(K; \mathbb{Z})$ is an Abelian group, its rank does not completely determine it. In particular,

$$H_q(K; \mathbb{Z}) \cong \underbrace{\mathbb{Z}^r}_{\textit{free part of } G} \oplus \underbrace{\mathbb{Z}_{p_1^{q_1}} \oplus \mathbb{Z}_{p_2^{q_2}} \oplus \ldots \oplus \mathbb{Z}_{p_k^{q_k}}}_{\textit{torsion of } G},$$

where the rank of the group $r = \mathfrak{b}_q(K;\mathbb{Z})$, referred to as the q-Betti number, only determines[54] the free part of the group.

Let $\operatorname{rank} \partial_q$ be the rank of the image of ∂_q. By Proposition 4.1, numbers $\mathfrak{b}_q$ can be deduced[55] from the ranks of $\partial_q, \partial_{q+1}$ and n_q. However, in order to compute torsion we need to delve deeper into the structure of the boundary maps.

For example, suppose the ranks of the two maps in the following diagram are 1:

$$\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z} \xrightarrow{\psi} \mathbb{Z},$$

and assume $\operatorname{Im} \varphi \subseteq \ker \psi$. Defining $H = \ker \psi / \operatorname{Im} \varphi$, we know that $\operatorname{rank} H = 0$. However, depending on maps φ, ψ group H could be any cyclic group. For example, if $\psi(n) = k \cdot s \cdot n$ for some $k, s \in \mathbb{N}$ and $\varphi(n) = k \cdot n$, then $H \cong \mathbb{Z}_s$.

In order to compute homology with coefficients on $\mathbb{Z}$ we may reduce each boundary matrix to its **Smith normal form**. Given a matrix with entries in $\mathbb{Z}$, its Smith normal form is:

$$D = \begin{pmatrix} a_1 & 0 & \cdots & \cdots & 0 \\ 0 & a_2 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & a_r & 0 \cdots 0 \\ \vdots & & & & 0 \cdots 0 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix},$$

where each diagonal entry a_i divides[56] the next one. The diagonal entries a_i are called **elementary divisors** and r is the rank[57] of the matrix.

Some properties of the Smith normal form for matrices with entries in $\mathbb{Z}$:

1. Each matrix with entries in $\mathbb{Z}$ has[58] a Smith normal form.
2. The form is obtained through the combination[59] of row reduction and the Euclidean algorithm for computing greatest common divisors.
3. The form is unique up to the signs of the elementary divisors.

The elementary divisors generate the torsion part of homology.

We now describe how to obtain homology groups using the Smith normal form.

- Choose $q \in \{0, 1, \ldots\}$.
- Assume matrix D above is the Smith normal form of ∂_{q+1} with all diagonal entries being positive.

[54] Two of the cases when homology group has no torsion:

- For any simplicial complex K we have $H_0(K;\mathbb{Z}) \cong \mathbb{Z}^{\mathfrak{b}_0}$, where $\mathfrak{b}_0$ is the number of components of K.
- If K is a planar graph then $H_1(K;\mathbb{Z}) \cong \mathbb{Z}^{\mathfrak{b}_1}$, where $\mathfrak{b}_1$ is the number of holes K generates in the plane.

[55] I.e., $\mathfrak{b}_p = n_p - \operatorname{rank} \partial_p - \operatorname{rank} \partial_{p+1}$.

☞ It turns out that amongst all possible choices of coefficients, homology with coefficients in $\mathbb{Z}$ contains the most information. Details of this statement are formalized in the Universal coefficient Theorem, which explains the connection between coefficients $\mathbb{Z}$ and all other coefficients.

[56] I.e., $a_i | a_{i+1}, \forall i \in \{1, 2, \ldots, r-1\}$.

[57] The rank of the matrix corresponding to a boundary map coincides with the rank of the boundary map.

[58] Formally, speaking, a Smith normal form of a matrix A with entries in $\mathbb{Z}$ is a factorization $A = UDV$, where D is a matrix of the mentioned form, and U and V are matrices with entries in $\mathbb{Z}$ with determinant ± 1. In particular, the last condition means that U and V are invertible, and that its inverses have entries in $\mathbb{Z}$.

[59] At this point the shortcoming of the structure of a group as compared to that of a field becomes prominent. When coefficients were in a field, we could always divide a row by a non-zero entry. When working with coefficients in $\mathbb{Z}$ that is not allowed (except for ± 1, which doesn't really help). As a result, obtaining the desired form of a matrix requires us to involve greatest common divisors and even then, not all non-trivial diagonal entries can be transformed to 1.

- Compute also the rank of ∂_q, Possibly also through the Smith normal form.
- Then:
$$H_q(K;\mathbb{Z}) \cong \mathbb{Z}^{n_q - \operatorname{rank}\partial_q - \operatorname{rank}\partial_{q+1}} \oplus \bigoplus_{i=1}^{r} \mathbb{Z}_{a_i}.$$
Note that this form may potentially be simplified[60] further.

We conclude by providing analogues of the examples of homology:

[60] If s_1, s_2 are relatively prime, then $\mathbb{Z}_{s_1 \cdot s_2} \cong \mathbb{Z}_{s_1} \oplus \mathbb{Z}_{s_2}$. Also, if some $a_i = 1$, then $\mathbb{Z}_{a_i}$ is the trivial group, i.e., it can be omitted from the expression.

- The formula for disjoint union holds as before: $H_i(K \coprod L;\mathbb{Z}) \cong H_i(K;\mathbb{Z}) \times H_i(K;\mathbb{Z})$.
- The expression of the Euler characteristic with integer Betti numbers holds by the same argument: $\chi(K) = \mathfrak{b}_0 - \mathfrak{b}_1 + \mathfrak{b}_2 - \mathfrak{b}_3 + \ldots$.
- For each $n \in \{1, 2, \ldots\}$ we have:
  - $H_0(S^n;\mathbb{Z}) \cong H_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$;
  - $H_i(S^n;\mathbb{Z}) = 0, \forall i \notin \{0, n\}$.
- For each connected manifold K of dimension n we have:
  - $H_n(K;\mathbb{Z}) = 0$ if K has boundary.
  - $H_n(K;\mathbb{Z}) \cong \mathbb{Z}$ if K is closed orientable.
  - $H_n(K;\mathbb{Z}) \cong \mathbb{Z}_2$ if K is closed non-orientable.
- If $\mathcal{K}$ is the Klein bottle, then $H_1(\mathcal{K};\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

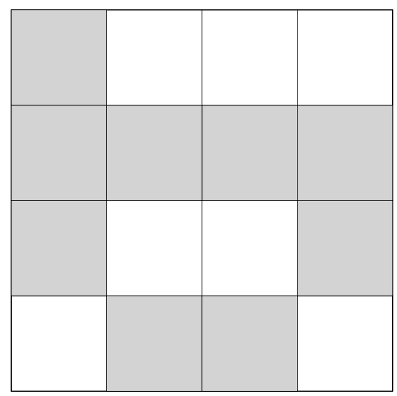

Figure 28: A 4×4 image consisting of grey pixels.

Appendix: The cubical homology

The homology construction we described above is called simplicial homology as it is based on the structure of a simplicial complex: a space assembled using simplices. However, there are settings in which alternative shapes of basic building blocks appear to be more suitable. One such setting is the image analysis, where we work with an image or a video consisting of pixels. In this setting it would be natural to consider pixels as the building blocks. This leads us to a new construction[61] of complexes and homology: cubical complexes and cubical homology. We will restrict ourselves to the setting of two-dimensional images, meaning the pixels are chosen from a fixed grid. The construction could easily be generalized to three-dimensional (movies of 2-D images or a 3-D image) four-dimensional (movies of 3-D images) or highere-dimensional images with different shapes of grids and cubes, or even without a fixed grid.

[61] Actually, we could build complexes and the corresponding theory from any shape of basic building blocks.

Let $n \in \mathbb{N}$ and consider a square grid of size $n \times n$, where n refers to the number of squares along each side, see Figure 28. Our image is given by a collection of pixels (grey squares). The first task is to define **cubical simplices**:

- 0-dimensional simplices are the vertices appearing on the grid. There are $(n+1)^2$ vertices.
- 1-dimensional simplices are the vertical and horizontal edges between vertices appearing on the grid. There are $2n(n+1)$ edges.
- 2-dimensional simplices are the squares of the grid. There are n^2 squares.

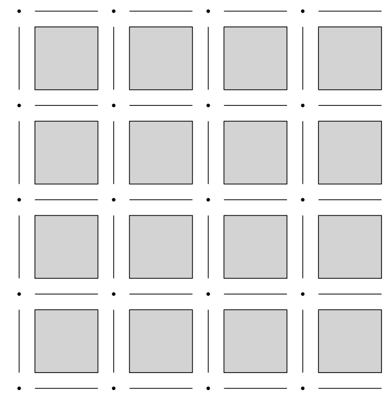

Figure 29: The collection of all potential cubical simplices.

A **cubical complex** K on an $n \times n$ grid is a collection of cubical simplices such that if $\sigma \in K$ and $\tau \subseteq \sigma$, then $\tau \in K$.

Our next task is to determine a convenient systematic labelling for the squares, edges and vertices. In the context of simplicial complexes the labels were just the oriented collections of vertices. While the same approach could[62] be used here, there is a more elegant enumeration of the simplices.

Instead of thinking about coordinates in terms of the $n \times n$ grid, we systematically imagine all potential simplices of a complex drawn in a table-like pattern as Figure 29 demonstrates. Each simplex can be assigned coordinates (x, y) where $x, y \in \{0, 1, 2, \ldots, 2n\}$ according to this pattern. Drawing the corresponding coordinate axes superimposed over the original $n \times n$ grid (Figure 30) we see that a pair of coordinates (x, y) represents the simplex[63], whose center is (x, y). We additionally define the orientations:

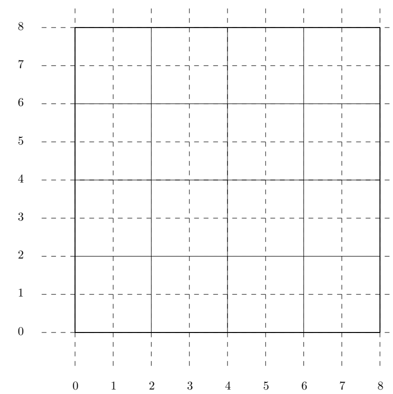

Figure 30: The assignment scheme.

[62] Although, the approach would be cumbersome. We would need 4 vertices to describe a square.

[63] A square, and edge, or a vertex.

- Each square is oriented with the ordering of its vertices in the positive-rotational order.
- Vertical edges are oriented upwards, horizontal to the right.

The resulting assignment of coordinates/labels has the following properties (see Figure 31):

- If x, y are both odd, then (x, y) is a square.
- If exactly one of x, y is odd, then (x, y) is an edge.
- If x, y are both even, then (x, y) is a vertex.

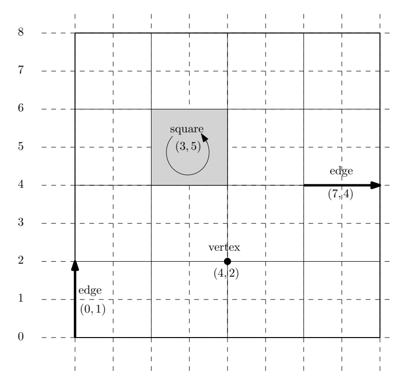

Figure 31: The assignment scheme.

We are now in a position to define the cubical homology. The structure of the definition is the same as for the simplicial homology with the only essential difference being the way in which we define the boundary map.

Let K be a cubical complex and choose[64] a a field of coefficients $\mathbb{F}$. For each $q \in \{0, 1, 2\}$ let n_q denote the number of cubical q-simplices in K.

[64] We could also choose the coefficients from an Abelian group, the construction would be analogous.

1. A q-chain is a formal sum $\sum_{i=1}^{n_q} a_i \sigma_i^q$ where $a_i \in \mathbb{F}$ and σ_i^q is an oriented cubical simplex of dimension q in K.

2. The chain group $\mathfrak{C}_q(K;\mathbb{F}) \cong \mathbb{F}^{n_q}$ is the vector space of all q-chains. Its generators are oriented cubical q-simplices of K.

3. For each $p \in \mathbb{N}$ the boundary map

$$\partial_p \colon \mathfrak{C}_p(K;\mathbb{F}) \to \mathfrak{C}_{p-1}(K;\mathbb{F})$$

is the linear map defined[65] by the following rules:

If x and y are both even (vertex): $\partial_p(x,y) = 0$.

If x is odd and y is even (horizontal edge):

$$\partial_p(x,y) = (x+1,y) - (x-1,y).$$

If x is even and y is odd (vertical edge):

$$\partial_p(x,y) = (x,y+1) - (x,y-1).$$

If x and y are both odd (square):

$$\partial_p(x,y) = (x+1,y) - (x,y+1) - (x-1,y) + (x,y-1).$$

As before, $\partial^2 = 0$.

4. The collection of chain groups bound together by the boundary homomorphisms is called the chain complex:

$$\cdots \xrightarrow{\partial} \mathfrak{C}_n(K;\mathbb{F}) \xrightarrow{\partial} \mathfrak{C}_{n-1}(K;\mathbb{F}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathfrak{C}_1(K;\mathbb{F}) \xrightarrow{\partial} \mathfrak{C}_0(K;\mathbb{F}) \xrightarrow{\partial} 0$$

5. For each $q \in \{0,1,\ldots\}$. We define groups:

- q-cycles as $\mathfrak{Z}_q(K;\mathbb{F}) = \ker \partial_q \le \mathfrak{C}_q(K;\mathbb{F})$.
- q-boundaries as $\mathfrak{B}_q(K;\mathbb{F}) = \operatorname{Im} \partial_{q+1} \le \mathfrak{Z}_q(K;\mathbb{F}) \le \mathfrak{C}_q(K;\mathbb{F})$.
- cubical q-homology group as the quotient

$$\mathfrak{H}_q(K;\mathbb{F}) = \mathfrak{Z}_q(K;\mathbb{Z})/\mathfrak{B}_q(K;\mathbb{F}).$$

It turns out that the cubical homology of a cubical simplex K is isomorphic to the homology of the union of the cubical simplices of K. In particular, the homology detects components, holes, and (in the case of higher dimensional cubical complexes) higher-dimensional holes as the usual homology would.

[65] The map encodes the geometric boundary.

⚠ The operations between the coordinates in the expression of the boundary map are the formal summations and subtractions of the chain group and should not be considered as operations of pairs. The coordinates (x,y) are only labels of cubical simplices and shouldn't be added to or subtracted from each other. For example, $(0,0)-(2,0)$ is a formal chain consisting of two vertices with coefficients 1 and -1, while label $(-2,0)$ is undefined.

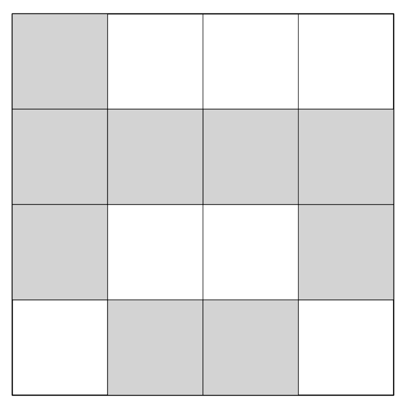

Figure 32: The cubical homology of the above image is given by $\mathfrak{H}_0 \cong \mathbb{F}$ (one component) and $\mathfrak{H}_1 \cong \mathbb{F}$ (one hole).