Simplicial complexes

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Topological and computational treatment of metric spaces relies on their convenient description. Given a metric space, we would like to have a finite combinatorial description, that can be used for computations. In the previous lectures we introduced planar triangulations as an example of such a description for planar subsets. In this lecture we will introduce simplicial complexes, which will form the basic structure upon which all our later computations will depend.

Simplicial complexes are higher-dimensional analogues of planar triangulations. While the later are collections of triangles that fit together nicely, simplicial complexes are collections of higher dimensional simplices (generalizations of triangles) that fit together nicely. Essentially we will be building spaces from simple building blocks (simplices) given a rule describing how these blocks fit together... just like LEGO cubes.

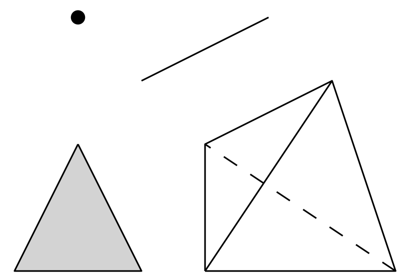

Figure 1: Some geometric simplices: a point, a line segment, a triangle, a tetrahedron.

1 Affine independence

A point, a line segment, a triangle, a tetrahedron, etc. These are some of the geometric simplices. They are basic building blocks of geometric simplicial complexes. A geometric simplex is a convex hull of a finite collection of points. Before we state their formal definition we need to clarify a general position property required of a set of points spanning such a simplex. Under this property we want a pair of points to span a line segment, a triple of points to span a triangle (and not just a line segment), etc.

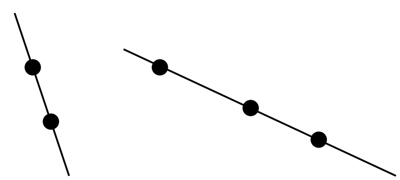

Figure 2: The affine hull of the two points on the left is a line. The affine hull of the three colinear points on the right is also a line, implying these three points are not affinely independent.

Choose $d, k \in \mathbb{N}$ and let $V = \{v_0, v_1, \ldots, v_k\} \subset \mathbb{R}^d$ be a collection of points. Their **affine combination** is any sum of the form

$$\sum_{i=0}^{k} \alpha_i v_i, \quad \text{with} \quad \sum_{i=0}^{k} \alpha_i = 1.$$

The **affine hull** of V is the collection of all affine combinations of elements of V. An affine hull is always an affine linear subspace in $\mathbb{R}^d$, meaning it is obtained from a linear subspaces of $\mathbb{R}^d$ by a translation.

Points $\{v_0, v_1, \ldots, v_k\}$ are **affinely independent** if no v_i can be expressed as an affine combination of points $V \setminus \{v_i\}$. Proposition 1.1 explains how to test points for affine independence using linear independence, and why each affine hull is a translated linear subspace.

Proposition 1.1. *Points of* $V = \{v_0, v_1, \ldots, v_k\} \subset \mathbb{R}^d$ *are affinely independent iff* $\{v_1 - v_0, v_2 - v_0, \ldots, v_k - v_0\}$ *are linearly indepen-*

dent.

Proof. Assume points of V are not affinely independent. Then, without the loss of generality, $v_0 = \sum_{i=1}^k \alpha_i v_i$ and $\sum_{i=1}^k \alpha_i = 1$, which implies equality $\sum_{i=1}^k \alpha_i(v_i - v_0) = 0$ and not all α_i are zero. We conclude that the points of V are not linearly independent.

On the other hand assume $\sum_{i=1}^k \beta_i(v_i - v_0) = 0$ with not all β_i being zero. We define $\beta_0 = -\sum_{i=1}^k \beta_i$ and observe that

$$\sum_{i=1}^k \beta_i v_i + \beta_0 v_0 = 0 \quad \text{and} \quad \sum_{i=0}^k \beta_i = 0.$$

Choose $K \in \{0, 1, \ldots, k\}$ so that $\beta_K \neq 0$. Then

$$v_K = \sum_{i=0,\ i\neq K}^k -\frac{\beta_i}{\beta_K} v_i \quad \text{and} \quad \sum_{i=0,\ i\neq K}^k -\frac{\beta_i}{\beta_K} = 1.$$

Hence points of V are not affinely independent. □

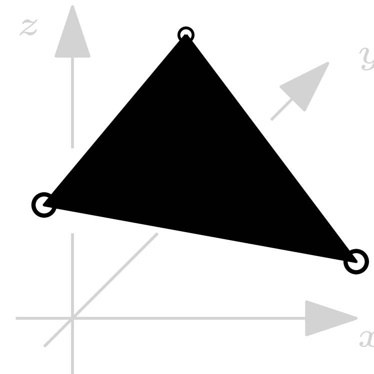

Figure 3: The convex hull of three affinely independent points is a triangle.

☞ A linearly independent collection of vectors in $\mathbb{R}^d$ can have at most d elements. An affinely independent collection of points in $\mathbb{R}^d$ can have at most $d+1$ elements.

Proposition 1.2. *Suppose points of $V = \{v_0, v_1, \ldots, v_k\} \subset \mathbb{R}^d$ are affinely independent. Then for each point $x \in \mathrm{Conv}(V)$ there exist unique coefficients $\alpha_i \in [0,1], i \in \{0, 1, \ldots, k\}$, such that*

$$x = \sum_{i=0}^k \alpha_i v_i \quad \text{and} \quad \sum_{i=0}^k \alpha_i = 1.$$

*Coefficients α_i in are called **barycentric coordinates** of point x in $\mathrm{Conv}(V)$.*

Proof. The existence of such coefficients α_i follows from $x \in \mathrm{Conv}(V)$. In order to prove the coefficients are unique assume the statement holds for two different sets of coefficients α_i and α_i', i.e.,

$$x = \sum_{i=0}^k \alpha_i v_i = \sum_{i=0}^k \alpha_i' v_i \quad \text{and} \quad \sum_{i=0}^k \alpha_i = \sum_{i=0}^k \alpha_i' = 1.$$

At some index i the coefficients α_i and α_i' differ. Without the loss of generality we can assume that index is zero, i.e., $\alpha_0 - \alpha_0' \neq 0$. Then

$$(\alpha_0 - \alpha_0')v_0 = \sum_{i=1}^k (\alpha_i' - \alpha_i) v_i$$

and

$$v_0 = \sum_{i=1}^k \frac{\alpha_i' - \alpha_i}{\alpha_0 - \alpha_0'} v_i,$$

which contradicts the assumption that the points of V are affinely independent. □

2 Geometric simplicial complex

We are now ready to define our basic building blocks.

Definition 2.1. *Let $k, d \in \{0, 1, \ldots\}$, $k \leq d$. A* ***geometric simplex*** *σ in $\mathbb{R}^d$ is the convex hull of an affinely independent family $V = \{v_0, v_1, \ldots, v_k\} \subset \mathbb{R}^d$, i.e., $\sigma = \mathrm{Conv}(V)$.*

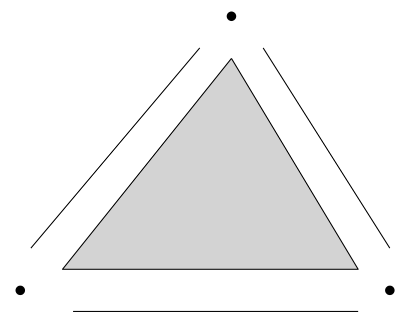

Figure 4: A two-dimensional simplex has six simplices as faces (three edges and three vertices), three of which are facets (edges).

The following is some terminology related to a geometric simplices $\sigma = \mathrm{Conv}(\{v_0, v_1, \ldots, v_k\})$:

- **Dimension**: $\dim(\sigma) = k$. We sometimes express it by writing it as supscript: $\sigma = \sigma^k$.
- **Vertices** of σ: $v_0, v_1, \ldots, v_k$.
- **Edges** of σ: convex hulls of pairs of vertices.
- We say that σ is **spanned** by the set of its vertices.
- If simplex τ is spanned by a subset of the vertices of σ, we say that:
  - τ is a **face** of σ.
  - σ is a **coface** of τ.
  - τ is a **facet** of σ if $\dim(\tau) = \dim(\sigma) - 1$.

Note that $\sigma^k \cong D^k$. By Proposition 1.2 each point of σ is uniquely described by its barycentric coordinates using the vertices of σ.

We can now use these building blocks to assemble more complicated spaces.

☞ All our simplicial complexes will be finite. For that reason we will be dropping the word "finite". There also exist simplicial complexes with infinitely many simplices. However, a proper definition of infinite simplicial complexes brings along additional technicalities which we want to avoid in our context.

Definition 2.2. *Let $d \in \{0, 1, \ldots\}$. A (finite)* ***geometric simplicial complex*** *$K \subset \mathbb{R}^d$ is a (finite) collection of geometric simplices, such that:*

a: If $\sigma \in K$ and τ is a face of σ, then $\tau \in K$.

b: If $\sigma, \tau \in K$, then $\sigma \cap \tau$ is either empty or a common face of both.

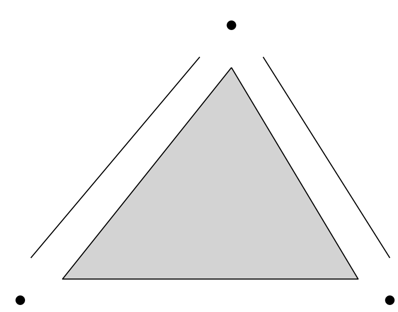

Figure 5: The smallest two-dimensional simplicial complex consists of a triangle and all its faces: three edges and three vertices.

Each planar triangulation has a corresponding simplicial complex consisting of all triangles, edges and vertices of the triangulation.

Let K be a simplicial complex. We define:

- **Dimension** $\dim(K) = \max_{\sigma \in K} \dim(\sigma)$. A one-dimensional simplicial complex is a graph.
- **Vertices** of K as the collection of all vertices of all simplices of K.

- **Edges** of K as the collection of all edges of all simplices of K.
- A geometric simplicial complex L is a **subcomplex** of K [notation: $L \leq K$], if $L \subset K$.
- For $n \in \{0, 1, \ldots\}$ the $n-$**skeleton** of K [notation: $K^{(n)}$] is the simplicial subcomplex of K consisting of all simplices of K of dimension at most n. For example, $K^{(0)}$ is the set of vertices of K.
- The **body** of K [notation: $|K|$] is the union of all simplices of K.

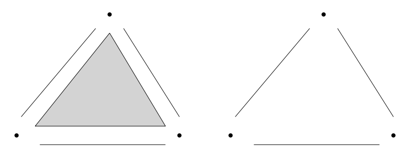

Figure 6: The simplicial complex from Figure 5 (left) and its 1-skeleton (right) consisting of three edges and three vertices.

Formally speaking, a geometric simplicial complex in $\mathbb{R}^d$ is a collection of simplices and its body is a subset in $\mathbb{R}^d$. In practice however we will be often identifying the two objects in geometric discussions. From now on we will be visualizing simplicial complexes by drawing their body and assuming the underlying structure of asimplicial complex.

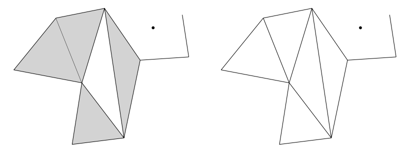

Figure 7: On the left there is a geometric simplicial complex presented by drawing its body. Each edge of a sketched triangle and each vertex of a sketched edge is assumed to be in the complex. On the right is the 1-skeleton of the simplicial complex on the left.

We are now ready to describe a connection between a metric subspace of a Euclidean space and its combinatorial description.

> **Definition 2.3.** *Let $d \in \{0, 1, \ldots\}$. A **triangulation** of a subspace $X \subset \mathbb{R}^d$ is a simplicial complex K in $\mathbb{R}^d$, such that $|K| \cong X$.*

Not every subspace of $\mathbb{R}^d$ admits a triangulation. However, all the subsets that will arise in our context will admit it. Triangulations of $B_{\mathbb{R}^2}((0,0),1)$ include examples in Figure 8 and Delaunay triangulations. A geometric simplicial complex is a triangulation of its body.

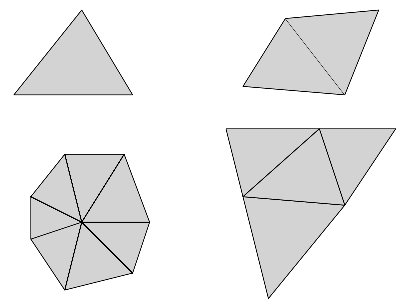

Figure 8: Some triangulations of D^2.

Occasionally we will want to refine the triangulation of a space, meaning we will want to decrease the size of simplices in order to improve visualisation, level of details, etc. Such refinements are called subdivisions. Given a geometric simplicial complex K, a geometric simplicial complex L is its **subdivision**, if each simplex of K is the union of a collection of simplices from L. As an example we already mentioned the barycentric subdivision of planar triangulations, which also exists for simplicial complexes. At this point we will refrain from introducing the formal definition. The lower right part of Figure 8 depicts a subdivision of a single 2-simplex, see also Figure 10.

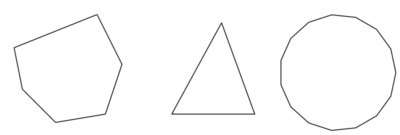

Figure 9: Some triangulations of S^1.

3 Abstract simplicial complex

When buying a commercial object to be assembled, be it a piece of furniture, a toy or a model made of cubes, or a picture made of puzzles, the package usually arrives in a big box. On the box is a picture of the object, which in our context represents the body of a geometric simplicial complex. On the picture we can often determine pieces, which in our setting would be geometric simplicial complexes.

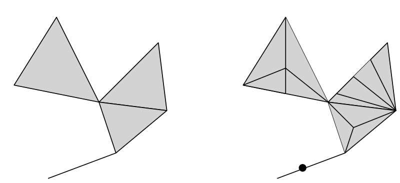

Figure 10: A simplicial complex and its subdivision.

Pieces on that picture have specific locations and just like geometric simplices, could be described by specific coordinates. However, the assembly instructions contain no coordinates. There is a good reason for that[1]. In order to assemble the object, the instructions only provide a list of pieces and instructions about how to put them together. That information is sufficient to reconstruct the object. Abstract simplicial complex plays a role of such instructions.

[1] Besides the fact that nobody would purchase such an item.

Assume we want to describe a geometric simplicial complex. That means we have to provide a list of all simplices. A simplex could be provided by a list of coordinates of its vertices, but then we also have to make sure the simplices intersect appropriately. It would be much easier to just list the simplices and describe how they fit together in a coordinate free way. Here is a way to do it.

> **Definition 3.1.** *Let V be a finite set. An* ***abstract simplicial complex*** *L on V is a family of non-empty subsets of V, such that if $\sigma \in L$ and $\tau \subseteq \sigma$ is non-empty, then $\tau \in L$.*

☞ In some sources the non-empty condition in the definition of an abstract simplicial complex is omitted and the empty set is always included as an abstract simplex of dimension -1.

A few more accompanying definitions using the notation of Definition 3.1:

- An **abstract simplex** σ is an element of L. Its **dimension** is $\dim(\sigma) = |\sigma| - 1$.
- If $\tau \subseteq \sigma \in L$, then:
  - τ is a **face** of σ.
  - σ is a **coface** of τ.
  - τ is a **facet** of σ if $\dim(\tau) + 1 = \dim(\sigma)$.
- **Dimension** $\dim(L)$ of L is the maximal dimension of a simplex in L.
- The (closed) **star** of a vertex $v \in K$ is $\mathrm{St}_K(v) = \mathrm{St}(v) = \{\sigma \in K \mid \sigma \cup \{v\} \in K\} \leq K$.
- The **link** of a vertex $v \in K$ is $\mathrm{Lk}_K(v) = \mathrm{Lk}(v) = \{\tau \in \mathrm{St}(v) \mid v \notin \tau\} \leq \mathrm{St}(v)$.

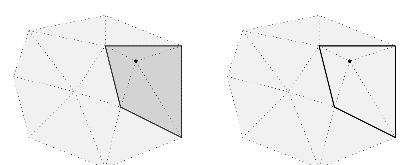

Figure 11: A star (left) and a link (right) of a vertex.

A geometric simplex is a subset of an Euclidean space, provided as the convex hull of the collection of its vertices. Each vertex is given as a point in space, usually in terms of coordinates. A geometric simplicial complex is a set of such simplices, contains all faces and has to satisfy the intersection properties of Definition 2.2.

An abstract simplex is just a collection of vertices. No coordinates are needed. An abstract simplicial complex is a set of such collections which contains all faces (all subsets of its elements). There are no intersections to be checked. It is a complete combinatorial description of a topological space.

Example 3.2. *Let K be a geometric simplicial complex provided by Figure 12. As a geometric simplicial complex, K contains specific geometric simplices described by the coordinates of their vertices. We can construct a corresponding abstract simplicial complex L. Label the vertices as demonstrated by the figure. Then*

$$L = \{\{a,c,d\},\{a,b\},\{b,c\},\{c,d\},\{d,a\},\{a,c\},\{a\},\{b\},\{c\},\{d\},\}.$$

No coordinates are involved. We could also only list the inclusion-maximal simplices, which completely determine the simplicial complex: $\{\{a,c,d\},\{a,b\},\{b,c\}\}$.

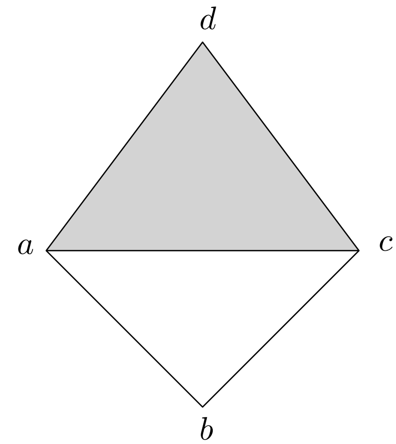

Figure 12: A picture accompanying Example 3.2.

A simpler structure of an abstract simplicial complex will suffice for most of our topological analysis of spaces and the corresponding computations. Indeed, it will simplify them. A geometric simplicial complex however is still useful when we want to visualise a complex. For example, outputs of various scans come in a form of geometric simplicial complexes modelling the scanned shape. While geometric simplicial complexes describe geometric information about the space (various sizes, lengths, etc.), abstract simplicial complexes contain only topological information (homeomorphic type).

It is easy to turn a geometric simplicial complex into an abstract simplicial complex: replace each coordinate given vertex by a unique label. The opposite is a bit harder. Turning an abstract simplicial complex into a geometric simplicial complex requires us to choose coordinates of vertices in line with the requirements for a geometric simplicial complex. If it can be done, such a geometric simplicial complex is called a **geometric realization** (or just realization) of the original abstract simplicial complex. It turns out that geometric realizations always exists, although obtaining them in a low-dimensional space is typically hard. The following are two special cases of such realizations.

Theorem 3.3. *Every abstract simplicial complex K with n vertices admits a geometric realization in $\mathbb{R}^{n-1}$.*

Proof. Simplicial complex K is a subcomplex of the full simplicial complex L on n vertices, i.e., the simplicial complex, whose simplices are all subsets of vertices of K. As L admits a realization in $\mathbb{R}^{n-1}$ as an $(n-1)$-simplex (i.e., the convex hull of a collection of n affinely independent points), so does K as its subspace. $\square$

Theorem 3.4. *Every abstract simplicial complex of dimension d admits a geometric realization in $\mathbb{R}^{2d+1}$.*

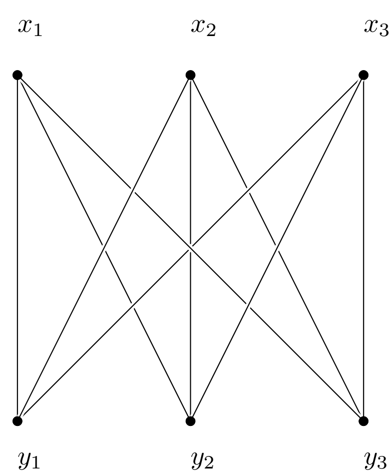

Figure 13: A sketch of a one-dimensional abstract simplicial complex (graph) with no realization in $\mathbb{R}^2$. The complex consists of all edges between x_i and y_j.

A proof of Theorem 3.4 is provided in Appendix.

As an example consider graphs, i.e., one-dimensional simplicial complexes. It is well known that some graphs are planar, which means they admit a geometric realization in the plane. However, there are graphs, that are not planar. These graphs can only be realized in $\mathbb{R}^3 = \mathbb{R}^{2\cdot 1+1}$ (and, of course, in $\mathbb{R}^m$ for $m > 3$). See Figure 13.

One of the goals of this course is the following: given an abstract simplicial complex, extract topological properties of its geometric realization. We will study and analyze spaces by working with their triangulations.

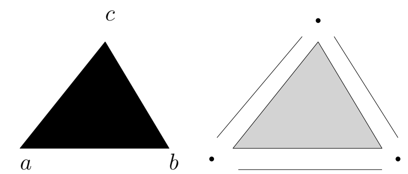

Figure 14: A metric space (left) and the corresponding geometric simplicial complex (right). The corresponding abstract simplicial complex is $\{\{a,b,c\},\{a,b\},\{b,c\},\{a,c\},\{a\},\{b\},\{c\}\}$.

Remark 3.5. *It is important to understand the differences between a metric space, its triangulation and a corresponding abstract simplicial complex. In practice however, we will be frequently vague in our expression for the sake of simplicity, ofter referring to just simplicial complex. Given a picture of a space like the one in Figure 12, we will keep in mind the three possible interpretations and use the one that fits the context at the moment.*

Also note that the nomenclatures are essentially the same for the abstract and geometric simplicial complexes. We declare this to be the case for all further[2] *definitions as well. For example, an abstract simplicial complex L is a triangulation of a metric space X if the corresponding geometric simplicial complex is, i.e., if the body of a geometric realization of L is homeomorphic to X.*

[2] Including the concepts of link and star of a complex at a point.

Example 3.6. *One of our standard examples of a metric space will be torus T. It is a two-dimensional metric space, actually a surface, depicted in Figure 15. A triangulation of T in terms of an abstract simplicial complex is provided by Figure 17.*

Topologically speaking, torus can be obtained from a square by identifying the opposite sides along the same direction. This construction is depicted in Figure 16.

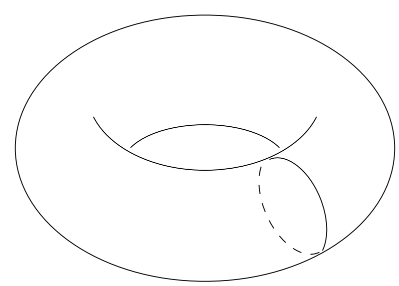
Figure 15: Torus.

As a result we can obtain a structure of an abstract simplicial complex by triangulating a square and respecting the mentioned identifications. This provides a convenient topological visualisation of a torus. Observe that a triangulation of T in terms of a geometric simplicial complex would be more complicated and not presentable in the plane.

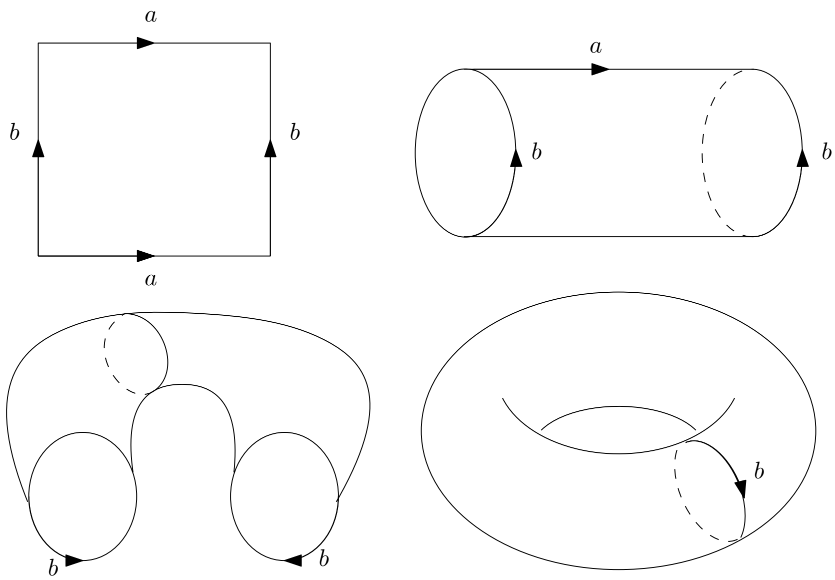

Figure 16: Torus arising from a square. Starting with a square (top left) we identify its pairs of opposite sides along the same direction as the labels and arrows suggest. Identifying the sides a we obtain a cylinder (top right). Identifying the other pair of sides, which represent loops b in the cylinder, we obtain the torus (bottom right).

The mentioned abstract simplicial complex is provided by Figure 17. We divide the square into 18 *triangles and keep in mind the identifications suggested by the arrows. The two sides of the square along the single arrows get identified along the direction of the arrows, and the same holds for the two sides along the double arrows. For the sake of clarity we also labeled the outer vertices of the triangles as each label appears at least twice due to the identifications.*

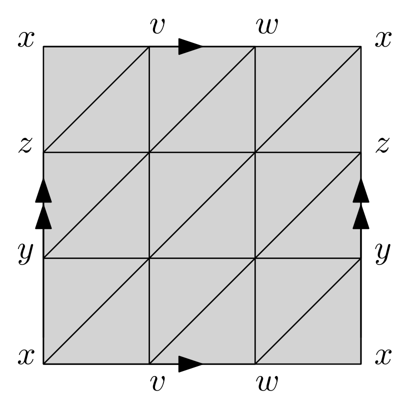

Figure 17: A triangulation of a torus in terms of an abstract simplicial complex.

Example 3.7. *Choose $n \in \{1, 2, \ldots\}$. In this example we provide the simplest triangulations of discs and spheres.*

Let σ^n be an n-simplex and define K to be the simplicial complex whose only maximal simplex is σ^n, i.e., K contains σ^n and all of its faces. Simplicial complex K is a triangulation of D^n.

To obtain a triangulation of S^{n-1} remove from K the maximal simplex, i.e., $K' = K \setminus \{\sigma^n\}$. Simplicial complex K consists of all faces of σ^n but does not contain σ^n itself. Simplicial complex K' is a triangulation of S^{n-1}.

Two invariants

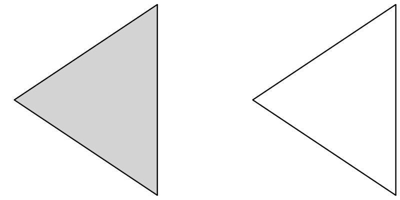

Figure 18: A triangle as a triangulation of D^2 and its boundary as a triangulation of S^1. In a similar way a solid tetrahedron is a triangulation of D^3 while its boundary is a triangulation of S^2.

Here we provide two invariants (of a space) that can be extracted from a triangulation. Both are homotopy invariants (and hence also topological invariants), meaning they coincide for homotopically equivalent spaces. A space typically has infinitely many possible triangulations. Imagine all possible Delaunay triangulations: they are all triangulations of D^2. We conclude that the numbers of vertices, edges, or higher dimensional simplices in a triangulation can't be topological invariants.

The first invariant is the number of components. Given a triangulation K, it is easy to extract that number from $K^{(1)}$ (which is a graph) using standard approaches of graph theory. Later we will explain in detail how to obtain this number in terms of homology. For example, the simplicial complex in Figure 19 has two components.

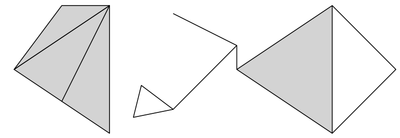

Figure 19: A complex with two components.

The second invariant is the Euler characteristic, which can be defined for simplicial complexes.

Definition 3.8. *Suppose K is a simplicial complex and let n_i denote the number of i-simplices in K. The* ***Euler characteristic*** $\chi(K) \in \mathbb{Z}$ *is defined as* $\chi(K) = n_0 - n_1 + n_2 - n_3 + \dots$.

The Euler characteristic of a metric space is the Euler characteristic of any of its triangulations.

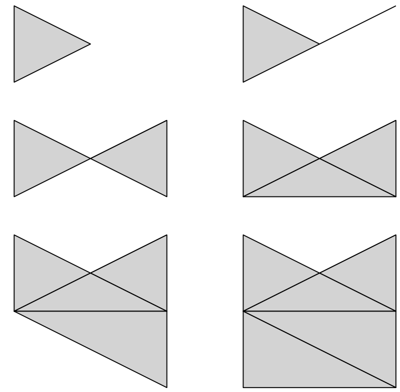

Figure 20: A series of homeomorphic simplicial complexes, each differing from the previous one by a small modifications. Note that the modifications preserve the Euler characteristic. In the first step we add a point and a vertex; in the second step a point we add a vertex, two edges and a triangle; and so on. Each new addition contributes a total of 0 to the Euler characteristic.

As was mentioned above, the Euler characteristic is homotopy invariant. Using this fact we can compute the following cases:

- Let X be a one-point space. Then $\chi(X) = 1$. Since each Delaunay triangulation K is homotopic to a point (meaning that $|K| \simeq X$), we also conclude $\chi(K) = 1$, a statement which we have already proved directly. In fact, the homotopy invariance implies that each triangulation of a contractible space is of Euler characteristic 1.
- The Euler characteristic of a torus is 0. It can be computed directly from the triangulation presented by Figure 17, which has 18 triangles, 27 edges and 9 vertices (keep in mind the identifications).
- Let $n \in \{0, 1, 2, \dots\}$. Then $\chi(S^n) = 1 + (-1)^n$.

  Proof. Let σ^{n+1} be an $(n+1)$-simplex and define K to be the simplicial complex whose only maximal simplex is σ^{n+1}. As we know K is a triangulation of D^{n+1}. As D^{n+1} is contractible, $\chi(K) = 1$. We also mentioned that $K' = K \setminus \sigma^{n+1}$ is a triangulation of S^n. As K' is obtained from K by removing an $(n+1)$-simplex, $\chi(K')$ is obtained from $\chi(K)$ by removing a contribution of that simplex, which is $(-1)^{n+1}$. Hence $\chi(S^n) = 1 - (-1)^{n+1} = 1 + (-1)^n$. □

☞ $\chi(S^n)$ could also be computed from the triangulation K' directly using the binomial formula.

4 Simplicial maps

Just as simplicial complexes provide a convenient combinatorial description of metric spaces, simplicial maps provide a combinatorial description of continuous maps. We first define them in the abstract setting.

Definition 4.1. *Suppose K and L are abstract simplicial complexes. A* ***simplicial map*** *between K and L is an assignment* $f\colon K^{(0)} \to L^{(0)}$ *on vertices, such that for each abstract simplex* $\{v_0, v_1, \dots, v_k\} \in K$ *its image* $\{f(v_0), f(v_1), \dots, f(v_k)\}$ *is an abstract simplex in L.*

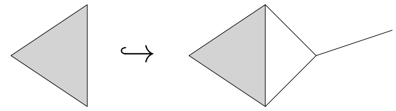

Figure 21: An embedding of a simplicial subcomplex $L \leq K$ into K is identity on the vertices and always a simplicial map.

Remark 4.2. *A simplicial map will be usually denoted by* $f\colon K \to L$*. However, since K and L are collections of sets, such a notation would formaly include maps that map, say, a vertex to an edge, a highly unfavourable occurence. When talking about simplicial maps* $K \to L$ *we thus always consider only maps in the sense of Definition 4.1, i.e., maps that map a vertex to a vertex, while images on simplices are always determined by the values on the vertices:*

- *For each vertex* $v \in K$ *we define a corresponding vertex* $f(v)$*.*
- *For each abstract simplex* $\sigma = \{v_0, v_1, \dots, v_k\} \in K$ *its image* $f(\sigma) = \{f(v_0), f(v_1), \dots, f(v_k)\}$ *is determined by the values on its vertices.*
- *Note that* $f(\sigma)$ *is a set, meaning there are no repetitions of elements. In particular this means that each vertex appears at most once in* $f(\sigma)$*, even if it appears multiple times as* $f(v_i)$*. As a result the image of an* n*-dimensional simplex can be of dimension less than or equal to* n*, but never more than* n*.*

Example 4.3. *Let K be the simplicial complex in Figure 22. Assignment* $a \mapsto a; \quad b \mapsto c; \quad c \mapsto c; \quad d \mapsto d; \quad e \mapsto b$ *can be verified to induce a simplicial map* $K \to K$*. Note that triangle* $\{a, b, c\}$ *gets mapped to edge* $\{a, c\}$*.*

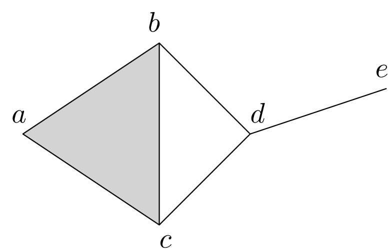

Figure 22: Simplicial complex K of example 4.3.

We are now ready to define simplicial maps in the geometric setting.

Definition 4.4. *Suppose K and L are geometric simplicial complexes. A map* $f\colon K \to L$ *is a* ***simplicial map****, if:*

1. *For each vertex* v *of K its image* $f(v)$ *is a vertex of L.*
2. *The corresponding map between the corresponding abstract simplicial complexes is simplicial, i.e., if* $\{v_0, v_1, \dots, v_k\}$ *span a geometric simplex in K then* $\{f(v_0), f(v_1), \dots, f(v_k)\}$ *span a geometric simplex in L.*
3. *Map* f *is linear on simplices (in terms of barycentric coordinates),*

i.e.,

$$\forall t_i \in [0,1], \quad \sum_{i=0}^{k} t_i = 1, \quad \forall v_i \in K^{(0)}: \quad f(\sum_{i=0}^{k} t_i v_i) = \sum_{i=0}^{k} t_i f(v_i).$$

Given a simplicial map between geometric simplicial complexes the induced map (i.e., the restriction to vertices) between abstract simplicial complexes is simplicial. Conversely, each simplicial map between abstract simplicial complexes corresponds to the unique simplicial map between the corresponding geometric simplicial complexes: the extension from vertices to geometric simplices is defined using the formula of item 3 of Definition 4.4. In accordance with our declarations simplicial maps will be used to denote maps either in geometric or abstract setting: in case of a preferred interpretation it will be stated explicitly or should be obvious from the context.

Simplicial maps between geometric simplicial complexes are continuous maps as they are linear (hence) continuous on each simplex. Surprisingly enough, each continuous map can be (up to homotopy) represented by a simplicial map, which means that as long as we are interested in homotopical properties, we can restrict ourselves to simplicial maps.

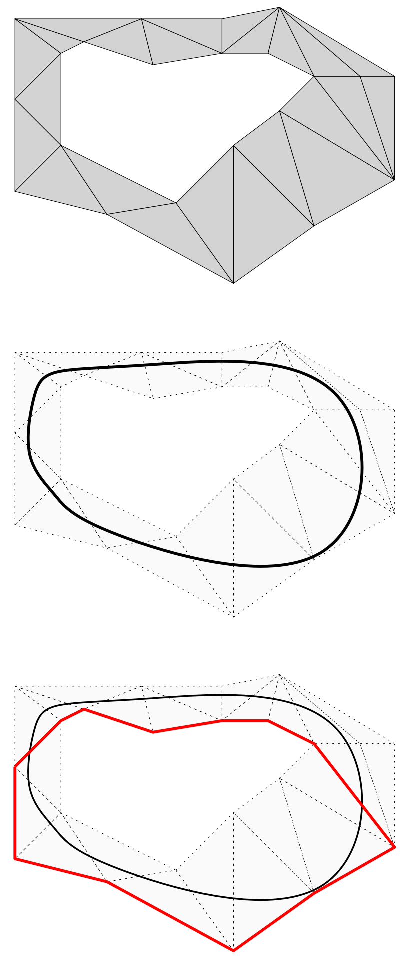

Figure 23: The upper part of the figure presents a simplicial complex. The middle part consists of a smooth curve (bold) superposed on the simplicial complex. The bottom part consists of a simplicial approximation (bold red) of a smooth curve superposed on the simplicial complex and the curve. The indicated simplicial approximation requires a triangulation of the domain S^1 with at least 12 edges.

Theorem 4.5. *Suppose $f\colon K \to L$ is a continuous map between geometric simplicial complexes. Then there exist sufficiently fine subdivisions K' of K and L' of L, and a simplicial map $f'\colon K' \to L'$, such that $f \simeq f'$. We call f' a* ***simplicial approximation*** *of f.*

The subdivisions above can be taken to be sufficiently fine barycentric subdivision. A continuous map between simplicial complexes is formally a map between the bodies of the simplicial complexes. In this sense both f and f' are maps $|K| = |K'| \to |L| = |L'|$ hence $f \simeq f'$ makes sense.

Elementary collapses

Elementary collapses are minor local modifications of simplicial complexes, which preserve its homotopy type. Conveniently enough, they can be described in purely combinatorial terms. Their importance stems from the following Lemma.

Lemma 4.6. *Let K be a geometric simplicial complex containing simplex $\sigma = \{v_0, v_1, \ldots, v_k\}$ and let $\tau = \{v_1, \ldots, v_k\}$ be its facet. If σ is the only coface*[3] *of τ, then the inclusion $i\colon K \setminus \{\tau, \sigma\} \hookrightarrow K$ is a homotopy equivalence.*

[3] i.e., a simplex which contains τ.

Proof. The proof is sketched in Figure 24.

We first need to subdivide σ and τ. Choose a point a in the middle of τ and connect it to all vertices of σ. This induces a subdivision of σ and τ and in fact of K as no other simplex contains[4] σ or τ.

In order to obtain a continuous deformation[5] from K to $K \setminus \{\tau, \sigma\}$, slide a towards v_0. This sliding is a linear homotopy and can be easily described in the barycentric coordinates of the new subdivision of K. $\square$

[4] If there was another simplex containing σ, then that simplex would have been a coface of τ, which contradicts our assumptions. In particular, If σ is the only coface, then τ is a facet of σ. This fact also refers to conditions of Definition 4.7.

[5] Homotopy equivalence.

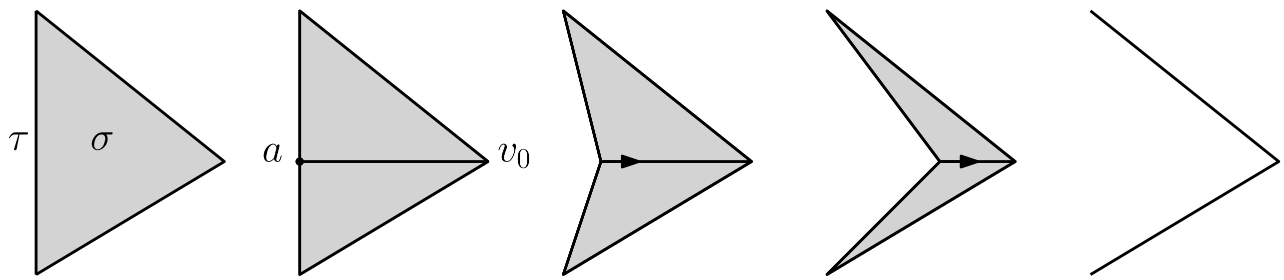

Figure 24: Elementary collapse from Lemma 4.6 with σ being the triangle and τ its left side.

Definition 4.7. *Let K be a simplicial complex, $\tau^{(k-1)} \subset \sigma^{(k)} \in K$, and assume σ is the only coface of τ. A removal $K \to K \setminus \{\tau, \sigma\}$ is called an **elementary collapse**.*

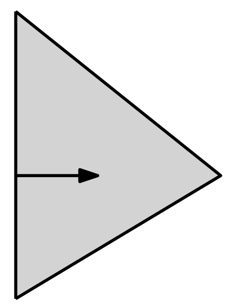

Figure 25: The elementary collapse of Figure 24 is usually indicated by an arrow from τ into σ.

By Lemma 4.6 each elementary collapse preserves the homotopy type of a complex. Note that the collapsing map $K \to K \setminus \{\tau, \sigma\}$ is not[6] a simplicial map on K. It is, however, a simplicial map on the subdivision of K employed in Lemma 4.6, defined by mapping $a \mapsto v_0$ and keeping all the other vertices intact. Its homotopy inverse is the inclusion $K \setminus \{\tau, \sigma\} \hookrightarrow K$, which is a simplicial map.

[6] Except if τ is a vertex.

Elementary collapses are convenient because they provide us with a simple combinatorial condition that can be used to induce homotopy equivalence on abstract simplicial complexes. This idea will be further expanded later within the context of Discrete Morse Theory.

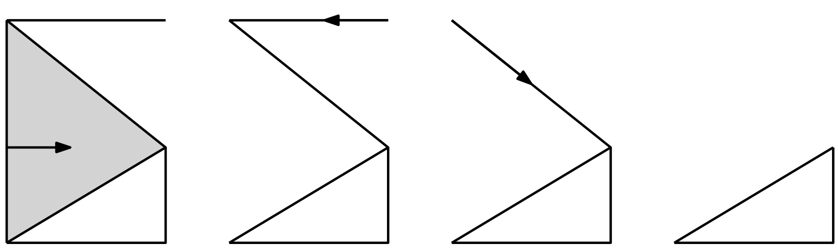

Figure 26: An example of a simplification of (the homotopy type of) a simplicial complex using elementary collapses.

5 Concluding remarks

Recap (highlights) of this chapter

- Geometric simplex;
- Geometric simplicial complex;
- Abstract simplicial complex;
- Geometric realization;
- Simplicial map;
- Simplicial approximation and elementary collapse.

Background and applications

Simplicial complexes model spaces in a wide spectrum of theory and applications. Great portions of topology and geometry are based on them due to their simple structure and amenability to combinatorial treatment. On the applied side simplicial complexes are typically used to model shapes.

Appendix: Proof of Theorem 3.4

Before we begin with the proof we clarify a fact that will be used. A generic[7] (random) collection of $n+1$ points in $\mathbb{R}^n$ is affinely independent. Geometrically this is easy to believe:

[7] Notion "generic" as used in this appendix is usually referred to as "general linear position" in the literature.

- Generic two points in $\mathbb{R}$ will be different;
- Generic three points in $\mathbb{R}^2$ will not be colinear;
- Generic four points in $\mathbb{R}^3$ will not be coplanar.

Similarly, even for $k > n+1$ a generic set V of k points in $\mathbb{R}^n$ has the same property: each a collection of $n+1$ points in V is affinely independent. For example, in a generic collection of points in the plane no triple of points will be collinear. We will only use the fact that generic collections exist. This fact can be proved using linear algebra.

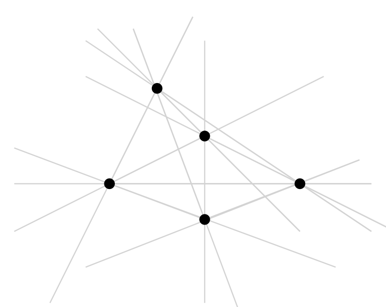

Figure 27: A generic collection of 5 points in the plane, meaning that no three are collinear. Only by adding a point on any grey line does the generic condition brake. If we add any other point to this collection, the obtained collection of 6 points is still generic.

Proof of Theorem 3.4. Let K be an abstract simplicial complex of dimension d, whose vertices are $v_0, v_1, \dots, v_k$. Choose a generic collection of points $V = \{x_0, x_1, \dots, x_k\} \subset \mathbb{R}^{2d+1}$, meaning that each collection of $2d+2$ points from V is affinely independent. We will prove that the correspondence $v_i \leftrightarrow x_i$ for all $i \in \{0, 1, \dots, k\}$ provides a geometric realization of K.

For each abstract simplex $\sigma \in K$ spanned by $v_{j_0}, v_{j_1}, \ldots, v_{j_m}$, the corresponding geometric simplex σ' is spanned by the affinely independent collection $x_{j_0}, x_{j_1}, \ldots, x_{j_m}$. It remains to prove that if $\sigma, \tau \in K$, then $\sigma' \cap \tau' = (\sigma \cap \tau)'$.

As $(\sigma \cap \tau)' \subseteq \sigma', \tau'$ by the definition, we have $\sigma' \cap \tau' \supseteq (\sigma \cap \tau)'$.

To prove the other inclusion we will make use of the dimension assumption. Let $z \in \sigma' \cap \tau'$, which means that z can be expressed as a convex combination of vertices in σ' and also as a convex combination of vertices in τ'. As the total number of vertices in τ' and σ' is at most $2n + 2$ (by the dimension assumption), the generic condition implies these are affinely independent, and thus the convex (affine) combinations above coincide as they have to be unique by Proposition 1.2. In particular, this means that only the barycentric coordinates corresponding to the vertices that lie in both simplices (i.e., $\sigma \cap \tau$) can be non-zero, which implies $z \in (\sigma \cap \tau)'$ and hence $\sigma' \cap \tau' \subseteq (\sigma \cap \tau)'$. □