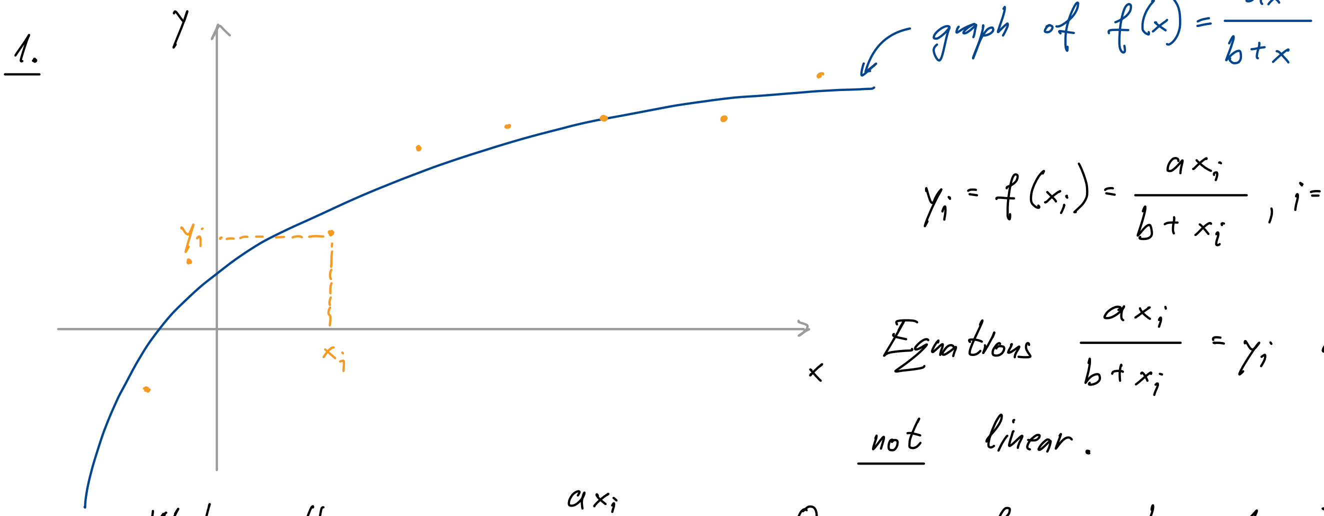
Matematično modeliranje (FRI), vaje, 23.03.2021

1.



$$y_i = f(x_i) = \frac{a x_i}{b + x_i}, \quad i = 1, \dots, 7$$

Equations $\frac{a x_i}{b + x_i} = y_i$ are not linear.

Write them as $\frac{a x_i}{b + x_i} - y_i = 0$, ie. for each $i = 1, .., 7$

we have

$$\frac{a x_1}{b + x_1} - y_1 = 0$$

$$\frac{a x_2}{b + x_2} - y_2 = 0$$

$$\vdots$$

$$\frac{a x_7}{b + x_7} - y_7 = 0$$

$$\vec{F} : \mathbb{R}^2 \longrightarrow \mathbb{R}^7$$

$$\vec{F}\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} \frac{a x_1}{b + x_1} - y_1 \\ \frac{a x_2}{b + x_2} - y_2 \\ \vdots \\ \frac{a x_7}{b + x_7} - y_7 \end{bmatrix}$$

If we had a perfect fit, we would just solve $\vec{F}\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \vec{0}$.

Or: find a minimum of $\begin{bmatrix} a \\ b \end{bmatrix} \longmapsto \left\| \vec{F}\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) \right\|^2$ $\leftarrow$ this is achieved by the nonlinear least squares method.

(a) First evaluate $\operatorname{grad}\left(" \begin{bmatrix} a \\ b \end{bmatrix} \longmapsto \underbrace{\left\| \vec{F}\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) \right\|^2}_{f(a,b)} " \right) = \operatorname{grad} f$

$$\frac{\partial f}{\partial a} = 2 \left\| \vec{F}\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) \right\| \cdot \frac{\partial \left\| \vec{F}\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) \right\|}{\partial a} = \dots = 2\left(\frac{a x_1}{b + x_1} - y_1\right) \cdot \frac{x_1}{b + x_1} + \dots$$

similar expressions, different indices

$$f(a,b) = \left\| \vec{F}\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) \right\|^2 = \vec{F}\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) \cdot \vec{F}\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \left(\frac{a x_1}{b + x_1} - y_1\right)^2 + \dots + \left(\frac{a x_7}{b + x_7} - y_7\right)^2$$

terms like first one with different indices

$$\frac{\partial f}{\partial b} = 2\left(\frac{a x_1}{b+x_1} - y_1\right)\cdot\frac{-a x_1}{(b+x_1)^2} + \dots$$

$\operatorname{grad} f = \vec{0}$ $\leftarrow$ to solve this using Newton's method we would need $\underbrace{J(\operatorname{grad} f)}_{H_f}$, where $\operatorname{grad} f = \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{bmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

Rest. H.W.

(b) To run the Gauss-Newton iteration we need (besides $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^7$) $J\vec{F}$.

$$\vec{F} = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_7 \end{bmatrix}, \quad J\vec{F} = \begin{bmatrix} \frac{\partial F_1}{\partial a} & \frac{\partial F_1}{\partial b} \\ \frac{\partial F_2}{\partial a} & \frac{\partial F_2}{\partial b} \\ \vdots & \vdots \\ \frac{\partial F_7}{\partial a} & \frac{\partial F_7}{\partial b} \end{bmatrix} = \begin{bmatrix} \frac{x_1}{b+x_1}, & -\frac{a x_1}{(b+x_1)^2} \\ \frac{x_2}{b+x_2}, & -\frac{a x_2}{(b+x_2)^2} \\ \vdots & \vdots \\ \frac{x_7}{b+x_7}, & -\frac{a x_7}{(b+x_7)^2} \end{bmatrix}$$

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2. $(x-p_i)^2 + (y-q_i)^2 = d_i^2$ .... $(x-p_i)^2 + (y-q_i)^2 - d_i^2 = 0$ for each i

number of transmitters

$$\vec{F}: \mathbb{R}^2 \longrightarrow \mathbb{R}^k \quad \dots \quad \vec{F}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \vdots \\ (x-p_i)^2 + (y-q_i)^2 - d_i^2 \\ \vdots \end{bmatrix}$$

$$J\vec{F}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \vdots & \vdots \\ 2(x-p_i), & 2(y-q_i) \\ \vdots & \vdots \end{bmatrix}$$

What about the initial guess? Use the "ad-hoc" solution from 2nd week.