1. We want to solve the equation

$$x = \log(x) + \frac{3}{2}.$$

   (a) Show that this equation has two solutions, one on the interval $[0, 1]$ and another on the interval $[2, 3]$.

   (b) Compute the solution on the interval $[2, 3,]$ using the fixed point method with the initial approximation $x_0 = 2$ to 2 decimal points of accuracy.

   (c) Compute the solution on the interval $[0, 1]$ using Newtons method with the initial approximation $x_0 = 0.5$ to 3 decimal points. Explain why it would not be possible to compute this solution using the fixed point method.

2. We want to compute Gaussian type integrals using the approximate formula

$$\int_0^\infty f(x)e^{-x^2/2}dx = af(0) + bf(c),$$

where a, b and c are variables that need to be determined. We know that

$$\begin{aligned} \int_0^\infty e^{-x^2/2}dx &= \sqrt{\frac{\pi}{2}} \\ \int_0^\infty xe^{-x^2/2}dx &= 1 \\ \int_0^\infty x^2e^{-x^2/2}dx &= \sqrt{\frac{\pi}{2}}. \end{aligned}$$

   (a) Write the conditions for a, b and c so that this formula will give the exact result for polynomials up to the order 2.

   (b) Solve the system of equations for a, b and c that you derived.

   (c) Using this formula, compute the integral

$$\int_0^\infty \sin(x)e^{-x^2/2}dx.$$

   What is the error, if you know this integral is approximately equal to 0.724778?

3. We are given a function $x(t)$ of the form

$$x(t) = a \cdot e^{bt}$$

Measurements have given the following results

t	100	125	150	175
x	9238	1724	323	63

   (a) By taking the logarithm and defining a new variable reduce the equation above to linear equation (in the new variables).

(b) Use the least square method to find the best approximation for a and b.

(c) What is the estimate of x given $t = 133$?

4. We want to solve the initial value problem

$$\begin{aligned} y'(x) &= e^{-x} - y(x) \\ y(0) &= 0 \end{aligned}$$

(a) Check that the general solution is

$$y(x) = e^{-x}(x + C)$$

(b) Determine the constant C to find the solution with $y(0) = 0$.

(c) Find the numerical approximation to the solution in the point $x = 1$ using the Runge-Kutta method of the 2. order with $h = 0.5$.

(d) What is the error of the approximation from (c)?

5. We are solving the initial value problem

$$\begin{aligned} y'(x) + 2xy(x) &= 0 \\ y(0) &= 1 \end{aligned}$$

(a) Verify that the general solution to the equation is

$$y(x) = e^{-x^2+C}$$

where C is a constant. Find the value of C so that the solution satisfies the initial condition.

(b) Using Eulers method with $h = 0.1$ find the approximate value of $x(0.5)$.

(c) What is the error of the approximation in (b)?

6. We want to numerically solve the equation

$$e^x - x - 2 = 0$$

(a) How many solutions does this equation have? Find their approximate location by sketching the functions.

(b) Determine the larger solution by regula falsi to 3 decimal places of accuracy.

(c) Determine the smaller solution using the fixed point method to 5 decimal places.

7. The following table gives the velocity of a car ($v[km/h\}$) in two seconds of driving

v	40	46	51	55	56
t	0	0.5	1	1.5	2

(a) Compute the distance the car has travelled using the trapeze rule.

(b) Compute the distance using Simpsons rule.

(c) What is (approximately) the acceleration at $t = 0.5$?

8. We would like to compute integral of f on the interval $[a, a+h]$ using the approximate formula

$$\int_a^{a+h} f(x)dx \approx \alpha f(a) + \beta f(a+h)$$

(a) Determine the coefficients α and β so that the formula is exact for polynomials up to degree 1.

(b) Using the formula form (a) compute the approximate value of

$$\int_0^{0.2} \sin(x)dx$$

(c) What is the relative error from (b)?

9. Measurements of a quantity f, that is dependant on time t, have given the table

t	0	1	2	3
$f(t)$	2.06	−0.99	−1.97	1.05

We would like to approximate $f(t)$ with a function of the form

$$f(t) = A\cos(\pi t/2) + B\sin(\pi t/2),$$

where A and B are constants we want to determine using the least square method

(a) Write the overdetermined system of equations for A and B.

(b) Write the normal system for A and B.

(c) Compute the solution to the normal system and estimate the value $f(4)$.

10. We are given a system of differential equations

$$\begin{aligned} x'(t) &= y(t) + 1 \\ y'(t) &= -x(t) \end{aligned}$$

with the initial values

$$\begin{aligned} x(0) &= 0 \\ y(0) &= 1. \end{aligned}$$

(a) Compute the apprixmate value of the solution at $t = 1$ using Eulers method with $h = 0.5$.

(b) Compute the apprixmate value of the solution at $t = 1$ using the 2. order Runge-Kutta method with $h = 0.5$.

(c) Compare the accuracy of both approximations if you know the exact solution is given by

$$(x(t), y(t)) = (2\sin(t), 2\cos(t) - 1).$$

11. We are solving the equation

$$x = e^x - 2$$

   (a) Show that this equation has solutions on the intervals $[-2, -1]$ and $[1, 2]$.

   (b) Compute the solution on the interval $[-2, -1]$ using the fixed point method to 3 decimal places.

   (c) Compute the solution on the interval $[1, 2]$ using Newtons method to 4 decimal places. Why can't we compute this solution using the fixed point method? (if we leave the equation on its current form).

12. Let $f(x) = \cos(x)$ and $g(x) = x$.

   (a) Show that the functions f and g intersect on the interval $[0, 1]$.

   (b) Using Newtons method find the intersection of f and g to 3 decimal places. For the initial value you can take $x = 0.5$.

   (c) Using (the basic) Simpson rule find the approximate area between the functions f and g between 0 and their intersection.

13. We are given some values of an unknown function f.

x	-2	-1	1	2
$f(x)$	18	2	4	12

We want to find a function g

$$g(x) = ax^2 + b,$$

that approximates f in the least square sense.

   (a) Write the overdetermined system of equations for the unknowns a and b.

   (b) Write the normal system for a and b

   (c) Solve the normal system and estimate the value $f(0)$.

14. We want to compute integrals on the the interval $[0, h]$ by the approximate formula

$$\int_0^h f(x)dx \approx \alpha f(0) + \beta f(2h/3)$$

   (a) Determine α and β so that the formula will be exact for polynomials up to the highest possible order.

(b) Compute the integral

$$\int_0^{\pi/2} \sin(x)dx$$

using the step size $h = \pi/8$

(c) Compute the integral above using the trapeze formula using the same step size. What are the error for both methods?

15. We are solving the equation $y'(x) = y(x)$ with $y(0) = 2$.

    (a) Find the numerical solution on the interval $[0, 1]$ using Eulers method with step size $h = 0.25$.

    (b) Find the numerical solution using the 2. order Runge-Kutta method with step size $h = 0.5$.

    (c) Find the exact solution.

    (d) What are the errors of the two numerical solutions at $x = 1$?

16. We are solving the equation

$$e^{-x} = \sin(x)$$

    (a) Sketch the graphs of the functions and approximately determine the locations of the solutions.

    (b) Using Newtons method find the first solution to 3 decimal points.

    (c) Find the second solution using the fixed point method to 3 decimal places. You must rewrite the equation in the form

$$x = \pi - \arcsin(e^{-x})$$

17. We are approximating the integral of f on the interval $[x_0, x_0 + h]$ with the formula

$$\int_{x_0}^{x_0+h} f(x)dx \approx \alpha f(c)$$

    where α and c are unknown values.

    (a) Determine α and c so that formula will be exact for polynomials up to the highest possible order.

    (b) Describe the composite rule for the integral $\int_a^b f(x)dx$ by subdividing the interval $[a, b]$ into n subinterval and using the rule from (a) for each subinterval.

    (c) Using the composite rule from (b) to compute the integral

$$\int_0^1 \sin(x)dx$$

    using 4 subintervals.

18. Approximate the data

x_i	0	$\pi/4$	$\pi/2$
y_i	1.1	1.5	0.9

by the function

$$f(x) = A\sin(x) + B\cos(x)$$

where A and B should be determined using the least square method.

(a) Write the overdetermined system for A and B.

(b) Write the normal system.

(c) Solve the normal system.

19. We are solving the initial value problem

$$\begin{aligned} y'(x) &= -2y(x) + e^{-x} \\ y(0) &= 1 \end{aligned}$$

(a) Find the numerical approximation of $y(0.4)$ using Eulers method with $h = 0.$.

(b) Find the exact solution if you know the general solution is of the form

$$y(x) = e^{-x}(Ae^{-x} + 1)$$

(c) Compute the error from (a).

20. The cross-section of a tunnel is given by the height y at location x

x	0	1	2	3	4	5	6
y	0	2	3	3.2	3	2	0

(a) Compute the area of the cross-section using the trapeze rule.

(b) Compute the area using Simpsons rule.

(c) What is the difference in the volume of the tunnel of depth 50 when we compare the results from (a) and (b)?