

**Linear least squares method.** Let  $A \in \mathbb{R}^{m \times n}$  be an  $m \times n$  matrix with  $m \geq n$ . Let  $\mathbf{b} \in \mathbb{R}^m$  be a vector. How would you find the orthogonal projection of  $\mathbf{b}$  on to the column space of  $A$ ,  $C(A)$ ? (Assume that the columns of  $A$  are linearly independent.)

1. We want to approximate a real function  $f$  on the interval  $[a, b]$  with a polynomial. We will do this (perhaps naïvely) by dividing the interval  $[a, b]$  with  $k + 1$  equidistant points  $a = x_0, x_1, \dots, x_k = b$  and then find the coefficients of the polynomial  $p(x)$  that is the best fit to the data in the table below according to the linear least squares method.

$x_0$	$x_1$	$\dots$	$x_i$	$\dots$	$x_k$
$f(x_0)$	$f(x_1)$	$\dots$	$f(x_i)$	$\dots$	$f(x_k)$

- (a) Let  $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  be a polynomial of degree  $n$ . Write the matrix  $A$  of the corresponding linear system and the right-hand side of  $\mathbf{b}$  according to the data in the table above.
  - (b) Find the approximations of orders 0, 1 and 2 for the function  $f(x) = \frac{x^2}{1+x^2}$  on the interval  $[-1, 1]$  using the points  $x_0 = -1$ ,  $x_1 = 0$  and  $x_2 = 1$ .
  - (c) Using octave approximate the function  $g(x) = \frac{1}{1+25x^2}$  on  $[-1, 1]$  with polynomials of order 0, 2, ..., 20, dividing the interval  $[-1, 1]$  with 21 equidistant points. Find the approximations for the exact data and for data with (artificially added) errors. Using the `plot` command plot the graphs of the original functions and all the approximations.
2. Use the linear least squares method to solve the following problem: In the plane  $\mathbb{R}^2$  we have  $n$  transmitters in known locations  $(p_1, q_1), \dots, (p_n, q_n)$ . A receiver can measure the distances  $d_1, \dots, d_n$  from these transmitters. Given those distances, we would like to determine the position of the receiver. In the ideal case the measurements are exact and for each  $i = 1, \dots, n$  we have an equation

$$(x - p_i)^2 + (y - q_i)^2 = d_i^2.$$

The solution of this system of equations then determines the unknown position  $(x, y)$ .

- (a) The first problem is that the equations are *not* linear. But the difference of two consecutive equations is linear. Write down these differences to obtain a system of  $n - 1$  linear equations.
- (b) Write the matrix  $A \in \mathbb{R}^{(n-1) \times 2}$  of the system and the corresponding right-hand side  $\mathbf{b} \in \mathbb{R}^{n-1}$ . Additional problem is that the measurements are not exact which means that the system  $A\mathbf{x} = \mathbf{b}$  (almost surely) has no solution.
- (c) Find the least squares solution to  $A\mathbf{x} = \mathbf{b}$ . Write an octave function `X = sprejemnik([pi, qi], [di])` that finds the position of the receiver  $X(x, y)$  given transmitter positions  $(p_i, q_i)$  and distances  $d_i$ . (The positions  $(p_i, q_i)$  are contained in an  $n \times 2$  matrix and the distances  $d_i$  are given in a column matrix of length  $n$ . The result  $X$  should be row vector  $X = [x, y]$ .)

- (d) Test the function using artificial data from the files `oddajniki.txt` and `razdalje.txt` found on *ucilnica*. You can import them into octave using the `load` command. Visualize the results.