Mathematical Modelling Exam

August 18th, 2022

This is an open book exam. You are allowed to use your notes, books and any other literature. You are NOT allowed to use any communication device. You have 100 minutes to solve the problems.

1. Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times r}$ and $C \in \mathbb{R}^{m\times r}$ be matrices. Consider the solutions of the matrix equations:
$$AXB = C. \tag{1}$$
Let $G_1 \in \mathbb{R}^{n\times m}$ and $G_2 \in \mathbb{R}^{r\times m}$ be generalized inverses of A and B, respectively.

   (a) Assume that $C = AG_1CG_2B$. Check that G_1CG_2 solves (1).

   (b) Prove that if (1) is solvable, then $C = AG_1CG_2B$ holds.
   Hint: Multiply (1) from left and from right by appropriate matrices and use the definitions of G_1, G_2.

   (c) Assume that (1) is solvable. Check that
   $$X = G_1CG_2 + Z - G_1AZBG_2$$
   solves (1) for any $Z \in \mathbb{R}^{n\times p}$.

Solution. First we check (1a):
$$A(G_1CG_2)B = AG_1CG_2B = C.$$

Second we prove (1b). We multiply (1) from left by AG_1 and from right by G_2B to obtain
$$AG_1AXBG_2B = AG_1CG_2B. \tag{2}$$
Since G_1 (resp. G_2) is a generalized inverse of A (resp. B), we have that $AG_1A = A$ (resp. $BG_2B = B$). Hence, (2) implies that
$$C = AXB = AG_1CG_2B, \tag{3}$$
where in the first equality we used (1). This proves (1b).

Finally we check (1c):
$$\begin{aligned} AXB &= A(G_1CG_2 + Z - G_1AZBG_2)B \\ &= AG_1CG_2B + A(Z - G_1AZBG_2)B \\ &= C + AZB - AG_1AZBG_2B \\ &= C + AZB - AZB = C, \end{aligned}$$
where we used (1b) in the second equality and the definitions of G_1, G_2 in the third equality.

Note: If (1) is solvable, then all solutions are of the form given in (1c). Indeed, if X solves (1), then $X = G_1CG_2 + X - G_1AXBG_2$, which means that $Z = X$ is one appropriate choice.

2. Let

$$\sqrt{\pi}\ln(x_1^2+x_2^2)-\frac{1}{\sqrt{\pi}}\sin(x_1x_2)=\ln(2\pi),$$
$$e^{x_1-x_2}+\frac{1}{\sqrt{\pi}}\cos(x_1x_2)=0,$$

be a nonlinear system and $v^{(0)}=\begin{bmatrix}\sqrt{\pi} & \sqrt{\pi}\end{bmatrix}^T$ a vector. Compute the approximation $v^{(1)}$ of the solution of the system using one step of Newton's method.

Solution. We define a vector function of a vector variable $F:\mathbb{R}^2\to\mathbb{R}^2$ by

$$F(x_1,x_2)=\begin{bmatrix}F_1(x_1,x_2)\\ F_2(x_1,x_2)\end{bmatrix}=\begin{bmatrix}\sqrt{\pi}\ln(x_1^2+x_2^2)-\frac{1}{\sqrt{\pi}}\sin(x_1x_2)-\ln(2\pi)\\ e^{x_1-x_2}+\frac{1}{\sqrt{\pi}}\cos(x_1x_2)\end{bmatrix}.$$

We are searching for the solution of $F(x_1,x_2)=0$ using Newton's method. We have that

$$v^{(1)}=v^{(0)}-\left(JF(v^{(0)})\right)^{-1}F(v^{(0)}),$$

where

$$\begin{aligned}JF(x_1,x_2)&=\begin{bmatrix}\frac{\partial F_1(x_1,x_2)}{\partial x_1} & \frac{\partial F_1(x_1,x_2)}{\partial x_2}\\ \frac{\partial F_2(x_1,x_2)}{\partial x_1} & \frac{\partial F_2(x_1,x_2)}{\partial x_2}\end{bmatrix}\\ &=\begin{bmatrix}\sqrt{\pi}\frac{2x_1}{x_1^2+x_2^2}-\frac{x_2}{\sqrt{\pi}}\cos(x_1x_2) & \sqrt{\pi}\frac{2x_2}{x_1^2+x_2^2}-\frac{x_1}{\sqrt{\pi}}\cos(x_1x_2)\\ e^{x_1-x_2}-\frac{x_2}{\sqrt{\pi}}\sin(x_1x_2) & -e^{x_1-x_2}-\frac{x_1}{\sqrt{\pi}}\sin(x_1x_2)\end{bmatrix}\end{aligned}$$

is the Jacobian matrix of F. So

$$JF(v^{(0)})=\begin{bmatrix}1-(-1) & 1-(-1)\\ 1 & -1\end{bmatrix}=\begin{bmatrix}2 & 2\\ 1 & -1\end{bmatrix}.$$

We compute $\left(JF(v^{(0)})\right)^{-1}$ using Gaussian elimination:

$$\begin{aligned}\underbrace{\left[\begin{array}{cc|cc}2 & 2 & 1 & 0\\ 1 & -1 & 0 & 1\end{array}\right]}_{\left[\begin{array}{c|c}JF(v^{(0)}) & I_2\end{array}\right]}\ &\underbrace{\sim}_{\substack{\ell_1=\frac{\ell_1}{2},\\ \ell_2=\ell_2-\frac{1}{2}\ell_1}}\left[\begin{array}{cc|cc}1 & 1 & \frac{1}{2} & 0\\ 0 & -2 & -\frac{1}{2} & 1\end{array}\right]\\ &\underbrace{\sim}_{\substack{\ell_2=-\frac{1}{2}\ell_2,\\ \ell_1=\ell_1+\frac{1}{2}\ell_2}}\underbrace{\left[\begin{array}{cc|cc}1 & 0 & \frac{1}{4} & \frac{1}{2}\\ 0 & 1 & \frac{1}{4} & -\frac{1}{2}\end{array}\right]}_{\left[\begin{array}{c|c}I_2 & \left(JF(v^{(0)})\right)^{-1}\end{array}\right]}.\end{aligned}$$

So

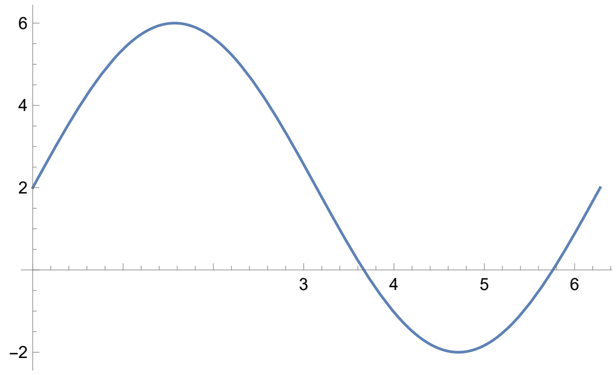
$$\begin{aligned}v^{(1)}&=\begin{bmatrix}\sqrt{\pi}\\ \sqrt{\pi}\end{bmatrix}-\begin{bmatrix}\frac{1}{4} & \frac{1}{2}\\ \frac{1}{4} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}\sqrt{\pi}\ln(2\pi)-\ln(2\pi)\\ 1-\frac{1}{\sqrt{\pi}}\end{bmatrix}\\ &=\begin{bmatrix}\sqrt{\pi}-\frac{1}{4}\sqrt{\pi}\ln(2\pi)+\frac{1}{4}\ln(2\pi)-\frac{1}{2}+\frac{1}{2\sqrt{\pi}}\\ \sqrt{\pi}-\frac{1}{4}\sqrt{\pi}\ln(2\pi)+\frac{1}{4}\ln(2\pi)+\frac{1}{2}-\frac{1}{2\sqrt{\pi}}\end{bmatrix}\\ &\approx\begin{bmatrix}1.20\\ 1.64\end{bmatrix}.\end{aligned}$$

3. Sketch the curve given in polar coordinates by

$$r(\varphi) = 2 + 4\sin(\varphi)$$

and compute the area of the smaller bounded region determined by the curve.

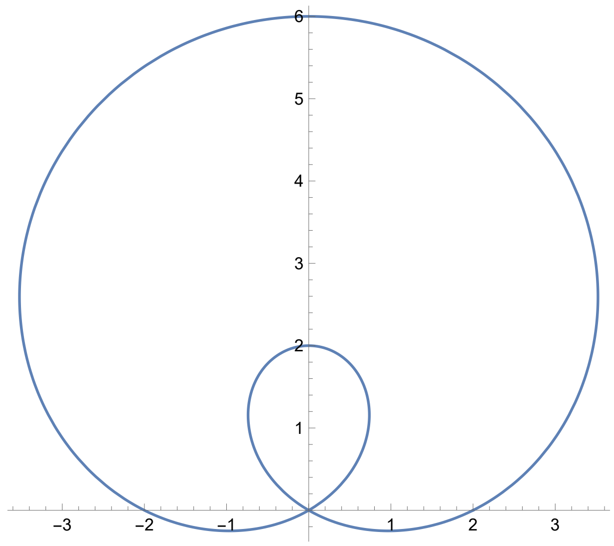
Solution. Since $r(\varphi)$ is periodic with a period 2π, we can restrict $r(\varphi)$ to the interval $[0, 2\pi]$. The sketch of $r(\varphi)$ is the following:



Let us write down $r(\varphi)$ for various φ:

φ	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	$\frac{6\pi}{6}$	$\frac{7\pi}{6}$	$\frac{8\pi}{6}$	$\frac{9\pi}{6}$	$\frac{10\pi}{6}$	$\frac{11\pi}{6}$
$r(\varphi)$	2	4	$\underbrace{2(1+\sqrt{3})}_{\approx 5.46}$	6	$\underbrace{2(1+\sqrt{3})}_{\approx 5.46}$	4	2	0	$\underbrace{2(1-\sqrt{3})}_{\approx -1.46}$	-2	$\underbrace{2(1-\sqrt{3})}_{\approx -1.46}$	0

Using this calculations we can sketch the curve:



We see from the sketch that the smaller bounded region enclosed by $r(\varphi)$ is obtained by restricting φ to the interval $[\frac{7\pi}{6}, \frac{11\pi}{6}]$. Its area A is

$$\begin{aligned}
A &= \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} r(\varphi)^2\, d\varphi = \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (2 + 4\sin\varphi)^2\, d\varphi \\
&= \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (4 + 16\sin\varphi + 16\sin^2\varphi)\, d\varphi \\
&= 2\,[\varphi]_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} + 8\,[-\cos\varphi]_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} + \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (8 - 8\cos(2\varphi))\, d\varphi \\
&= \frac{4\pi}{3} - 8\sqrt{3} + 4\,[\varphi]_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} - 2\,[\sin(2\varphi)]_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} = \frac{12\pi}{3} - 8\sqrt{3},
\end{aligned}$$

where we used that $\sin^2\varphi = \frac{1}{2}(1-\cos(2\varphi))$ in the fourth equality.

4. Solve the differential equation

$$\ddot{x} - \dot{x} - 4x = 2t + e^t. \tag{4}$$

Solution. First we solve the homogenenous part

$$\ddot{x} - \dot{x} - 4x = 0.$$

The characterstic polynomial is $p(\lambda) := \lambda^2 - \lambda - 4$ and hence

$$p(\lambda) = 0 \quad \Leftrightarrow \quad \lambda_{1,2} = \frac{1 \pm \sqrt{1+16}}{2} = \frac{1 \pm \sqrt{17}}{2}.$$

So, the solution of the homogeneous part is

$$x_h(t) = Ae^{\frac{1+\sqrt{17}}{2}t} + Be^{\frac{1-\sqrt{17}}{2}t},$$

where $A, B \in \mathbb{R}$ are constants. To find one particular solution we use the form

$$x_p = x_{p_1}(t) + x_{p_2}(t),$$

where

$$x_{p_1}(t) = Ct + D, \qquad x_{p_2}(t) = Ee^t.$$

Hence, $\dot{x}_{p_1} = C$, $\ddot{x}_p = 0$ and $\dot{x}_{p_2} = \ddot{x}_{p_2} = Ee^t$. Plugging this into the DE we obtain

$$Ee^t - (C + Ee^t) - 4(Ct + D + Ee^t) = 2t + e^t. \tag{5}$$

Comparing the coefficients at $1, t, e^t$ on both sides of (5) we get the system

$$-C - 4D = 0, \qquad -4C = 2, \qquad -4E = 1,$$

with the solution

$$D = \frac{1}{8}, \quad C = -\frac{1}{2}, \quad E = -\frac{1}{4}.$$

Hence, the general solution of the DE is

$$x(t) = Ae^{\frac{1+\sqrt{17}}{2}t} + Be^{\frac{1-\sqrt{17}}{2}t} - \frac{1}{2}t + \frac{1}{8} - \frac{1}{4}e^t.$$