

Mathematical Modelling Exam

June 3rd, 2022

This is an open book exam. You are allowed to use your notes, books and any other literature. You are NOT allowed to use any communication device. You have 100 minutes to solve the problems.

1. Let

$$A = \begin{bmatrix} -2 & 2 \\ -3 & -2 \\ -2 & -3 \end{bmatrix}.$$

- Find the matrix $B \in \mathbb{R}^{3 \times 2}$ of rank 1, which is the closest to A in the Frobenius norm.
- Calculate $\|A - B\|_F$.

2. Let

$$\begin{aligned} x^2 + y &= 37, \\ x - y^2 &= 5, \\ x + y + z &= 3 \end{aligned}$$

be a nonlinear system and $v^{(0)} = [0 \ 0 \ 0]^T$ a vector.

- Compute the approximation $v^{(1)}$ of the solution of the system using one step of Newton's method.
- Compute the tangent plane to the surface given by the equation $z = f(x, y)$, where

$$f(x, y) = 8xy + 4,$$

in the point $(1, 1)$.

3. Let

$$f(t) = (\sin t, \cos(3t)), \quad t \in [0, 2\pi]$$

be the parametric curve.

- Find all points where the curve intersects the coordinate axes.
- Find all points on the curve where the tangent is horizontal or vertical.
- Sketch the curve.

4. Consider the system of nonlinear differential equations

$$\begin{aligned} \dot{x} &= x(3 - x - 2y), \\ \dot{y} &= y(4 - 3x - y). \end{aligned}$$

- Find the stationary points of the system.
- Compute the linearization of the system around the nontrivial stationary point, i.e., the one with both coordinates being nonzero.
- Solve the linear system from the previous question and sketch its phase portrait.