Mathematical modelling

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Faculty of Computer and Information Science
University of Ljubljana

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Test for linear independence of solutions

Let $x_1(t), \ldots, x_n(t)$ be the solutions of the homogeneous part of (**??**) and form a matrix

$$W(x_1(t), \ldots, x_n(t)) := \begin{bmatrix} x_1(t) & \ldots & x_n(t) \\ \dot{x}_1(t) & \ldots & \dot{x}_n(t) \\ \vdots & \ddots & \vdots \\ x_1^{(n-1)}(t) & \ldots & x_n^{(n-1)}(t) \end{bmatrix}$$

We call the determinant

$$\phi(t) = \det W(x_1(t), \ldots, x_n(t)) : I \to \mathbb{R}$$

the Wronskian determinant of $W(x_1(t), \ldots, x_n(t))$, where I is the interval on which t lives.

Theorem (Existence and uniqueness of solutions)

If $x_1(t), \ldots, x_n(t)$ are solutions of a LDE with continuous coefficient functions $a_1(t), \ldots a_n(t)$, then their Wronskian is either identically equal to 0 or nonzero at every point. In other words, if $W(x_1, \ldots, x_n)$ has a zero at some point t_0, then it is identically equal to 0.

Proof of theorem

Let π_n be the set of all permutations of the set $\{1, \ldots, n\}$. Now we differentiate $\phi(t)$ and obtain

$$
\begin{aligned}
\phi'(t) &= \left(\sum_{\sigma \in \pi_n} x_{\sigma(1)} x^{(1)}_{\sigma(2)} \cdots x^{(n-1)}_{\sigma(n)} \right)' (t) \\
&= \sum_{\sigma \in \pi_n} \Big(\left(x_{\sigma(1)}\right)' (t) x^{(1)}_{\sigma(2)}(t) \cdots x^{(n-1)}_{\sigma(n)}(t) \\
&\qquad + x_{\sigma(1)}(t) \left(x^{(1)}_{\sigma(2)} \right)' (t) \cdots x^{(n-1)}_{\sigma(n)}(t) + \cdots \\
&\qquad + x_{\sigma(1)}(t) x^{(1)}_{\sigma(2)}(t) \cdots \left(x^{(n-1)}_{\sigma(n)}(t) \right)' \Big) \\
&= \left(\sum_{\sigma \in \pi_n} x^{(1)}_{\sigma(1)}(t) x^{(1)}_{\sigma(2)}(t) \cdots x^{(n-1)}_{\sigma(n)}(t) \right) + \\
&\qquad \left(\sum_{\sigma \in \pi_n} x_{\sigma(1)}(t) x^{(2)}_{\sigma(2)}(t) x^{(2)}_{\sigma(3)}(t) \cdots x^{(n-1)}_{\sigma(n)}(t) \right) + \\
&\qquad \cdots + \left(\sum_{\sigma \in \pi_n} x_{\sigma(1)}(t) x^{(1)}_{\sigma(2)}(t) \cdots x^{(n-2)}_{\sigma(n)}(t) x^{(n)}_{\sigma(n)}(t) \right).
\end{aligned}
$$

Now notice that the first $n-1$ summand are the determinants of the matrices

$$\begin{bmatrix} x_1(t) & \dots & x_n(t) \\ \vdots & \ddots & \vdots \\ x_1^{(i)}(t) & \dots & x_n^{(i)}(t) \\ x_1^{(i)}(t) & \dots & x_n^{(i)}(t) \\ \vdots & \ddots & \vdots \\ x_1^{(n-1)}(t) & \dots & x_n^{(n-1)}(t) \end{bmatrix} \tag{1}$$

and hence are equal to 0.
For the last summand use the initial DE (**??**) to express

$$x_{\sigma(n)}^{(n)} := -a_{n-1}(t)x_{\sigma(n)}^{(n-1)} - \dots - a_0(t)x_{\sigma(n)}.$$

The summands of the from $-a_i(t)x_{\sigma(n)}^{(i)}$ for $i < n-1$ give 0 terms in the sum $\sum_{\sigma \in \pi_n}$ since the sum is just the $-a_i(t)$ multiple of the determinant of the form (1), while the term $-a_{n-1}(t)x_{\sigma(n)}^{(n-1)}$ gives

$$-a_{n-1}(t)\phi(t).$$

It follows that $\phi(t)$ satisfies the DE

$$\phi'(t) = -a_{n-1}(t)\phi(t).$$

The theorem follows by noticing that the solution of this DE is

$$\phi(t) = ke^{-\int a_{n-1}(t)dt}, \quad \text{where} \quad k \in \mathbb{R}.$$

Second order homogeneous LDE with constant coefficients

We are given a DE

$$a\ddot{x} + b\dot{x} + cx = 0,$$

where $a, b, c \in \mathbb{R}$ are real numbers. We know from the theory above that the general solution is

$$x(t, C_1, C_2) = C_1 x_1(t) + C_2 x_2(t),$$

where $C_1, C_2 \in \mathbb{R}$ are parameters and

1. $x_1(t) = e^{\lambda_1 t}$ and $x_2(t) = e^{\lambda_2 t}$ if the characteristic polynomial has two distinct real roots,
2. $x_1(t) = e^{\alpha t} \cos \beta t$ and $x_2(t) = e^{\alpha t} \sin \beta t$ if the characteristic polynomial has a complex pair $\lambda_{12} = \alpha \pm i\beta$ of roots, and
3. $x_1(t) = e^{\lambda t}$, $x_2(t) = te^{\lambda t}$ if the characteristic polynomial has one double real root.

Nonhomogeneous LDEs

We are given the nonhomogeneous LDE

$$x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = f(t),$$

where $f : I \to \mathbb{R}$ is a nonzero function on the interval I. The following holds:

- If x_1 and x_2 are solutions of the nonhomogeneous equation, the difference $x_1 - x_2$ is a solution of the corresponding homogeneous equation.
- The general solution is a sum

$$x(t, C_1, C_2) = x_p + x_h = x_p + C_1x_1 + \cdots + C_nx_n,$$

  where x_p is a particular solution of the nonhomogeneous equation and $x_1, \ldots, x_n$ are linearly independent solutions of the homogeneous equation.
- The particular solution can be obtained using the method of "intelligent guessing" or the method of variation of constants.

The method of "intelligent guessing" typically works if the function $f(t)$ belongs to a class of functions that is closed under derivations, like polynomials, exponential functions and sums of these.

Example ($\ddot{x} + \dot{x} + x = t^2$)

We are guessing that the particular solution will be of the form

$$x_p(t) = At^2 + Bt + C.$$

We have that

$$\dot{x}_p(t) = 2At + B, \quad \ddot{x}_p(t) = 2A,$$

and so

$$\begin{aligned}\ddot{x} + \dot{x} + x &= 2A + (2At + B) + (At^2 + Bt + C) \\ &= At^2 + (2A + B)t + (2A + B + C)\end{aligned}$$

The initial DE gives us a linear system in A, B, C :

$$A = 1, \quad 2A + B = 0, \quad 2A + B + C = 0$$

with the solution $A = 1, B = -2, C = 0$. Hence, $x_p(t) = t^2 - 2t$.

Example ($\ddot{x} - 3\dot{x} + 2x = e^{3t}$)

We are guessing that the particular solution will be of the form

$$x_p(t) = Ae^{3t}.$$

We have that

$$\dot{x}_p(t) = 3Ae^{3t}, \quad \ddot{x}_p(t) = 9Ae^{3t},$$

and so

$$\ddot{x} - 3\dot{x} + 2x = 9Ae^{3t} - 3(3Ae^{3t}) + 2Ae^{3t} = 2Ae^{3t}$$

The initial DE gives us an equation $2A = 1$ and hence, $x_p(t) = \frac{1}{2}e^{3t}$.

Example ($\ddot{x} - x = e^t$)

The particular solution will not be of the form $x_p(t) = Ae^t$, since this is a solution of the homogeneous equation, we are guessing that the correct form in this case is

$$x_p(t) = Ate^t.$$

We have that

$$\dot{x}_p(t) = A(e^t + te^t), \quad \ddot{x}_p(t) = A(2e^t + te^t),$$

and so

$$\ddot{x} - x = A(2e^t + te^t) - Ate^t = 2Ae^t.$$

The initial DE gives us an equation $2A = 1$ and hence, $x_p(t) = \frac{1}{2}te^t$.

Example ($\ddot{x} + x = \frac{1}{\cos t}$)

Let us first solve the homogeneous part $\ddot{x} + x = 0$. The characteristic polynomial is $p(\lambda) = \lambda^2 + 1$ with zeroes

$$\lambda_{1,2} = \pm i = \cos t \pm i \sin t.$$

Hence, real solutions of the DE are

$$x_1(t) = \cos t \qquad \text{and} \qquad x_2(t) = \sin t. \tag{2}$$

So the general solution to the homogeneous part is

$$x(t) = C_1 x_1(t) + C_2 x_2(t), \qquad \text{where} \quad C_1, C_2 \in \mathbb{R} \quad \text{are constants.}$$

Now we are searching for the particular solution $x_p(t)$ of the form

$$x_p(t) = C_1(t) x_1(t) + C_2(t) x_2(t).$$

Thus,

$$\dot{x}_p(t) = \dot{C}_1(t)x_1(t) + C_1(t)\dot{x}_1(t) + \dot{C}_2(t)x_2(t) + C_2(t)\dot{x}_2(t). \tag{3}$$

We force an equation

$$\dot{C}_1(t)x_1(t) + \dot{C}_2(t)x_2(t) = 0. \tag{4}$$

Differentiang (3) further under the assumption (4) we get

$$\ddot{x}_p(t) = (\dot{C}_1(t)\dot{x}_1(t) + C_1(t)\ddot{x}_1(t)) + (\dot{C}_2(t)\dot{x}_2(t) + C_2(t)\ddot{x}_2(t)). \tag{5}$$

Plugging this into the initial DE and using that x_1, x_2 are solutions of $\ddot{x} + x = 0$

$$\dot{C}_1(t)\dot{x}_1(t) + \dot{C}_2(t)\dot{x}_2(t) = \frac{1}{\cos t}. \tag{6}$$

Expressing $\dot{C}_2(t)$ from (4) and plugging into (6) we get

$$\dot{C}_1(t)\dot{x}_1(t) - \frac{\dot{C}_1(t)x_1(t)}{x_2(t)}\dot{x}_2(t) = \dot{C}_1(t)\frac{\dot{x}_1(t)x_2(t) - x_1(t)\dot{x}_2(t)}{x_2(t)} = \frac{1}{\cos t}. \tag{7}$$

Using (2) in (7) we get

$$\dot{C}_1(t) = -\frac{\sin t}{\cos t}. \tag{8}$$

Hence,

$$C_1(t) = -\int \frac{\sin t}{\cos t} dt = -\int \frac{1}{u} du = -\log|u| = -\log|\cos t|,$$

where we used the substitution $u = \cos t$.

Using (8) in (4) we get

$$\dot{C}_2(t) = 1. \tag{9}$$

Hence,

$$C_2(t) = t.$$

So,

$$x_p(t) = -\log|\cos t| \cdot \cos t + t \sin t.$$

The complete solution to DE is

$$x(t) = C_1 \cos t + C_2 \sin t - \log|\cos t| \cdot \cos t + t \sin t,$$

where C_1, C_2 are parameters.

Vibrating systems

There are many vibrating systems in many different domains. The mathematical model is always the same, though. We will have in mind a vibrating mass attached to a spring.

Case 1: Free vibrations without damping

Let $x(t)$ denote the displacement of the mass from the equillibrium position.

- According to **Newton's second law of motion**

$$m\ddot{x} = \sum F_i,$$

  where F_i are forces acting on the mass.
- By **Hooke's law**, the only force acting on the mass pulls towards the equilibrium, its size is proportional to the displacement and the direction is opposite

$$F = -kx(t), \quad k > 0.$$

- So the DE in this case is

$$\boxed{m\ddot{x} + kx = 0}.$$

- The characteristic equation

$$m\lambda^2 + k = 0$$

  has complex solutions $\lambda = \pm\omega i$, $\omega^2 = k/m$.
- The general solution is

$$x(t) = C_1 \cos \omega t + C_2 \sin \omega t.$$

- So the solutions $x(t)$ are periodic. The equillibrium point $(0, 0)$ in the phase plane (x, v) is a center.

Case 2: Free vibrations with damping

We assume a linear damping force

$$F_d = -\beta \dot{x},$$

so the DE is

$$\boxed{m\ddot{x} + \beta \dot{x} + kx = 0}, \quad \text{where} \quad m, \beta, k > 0.$$

Depending on the solutions of the characteristic equation there are three cases:

- Overdamping when $D = \beta^2 - 4km > 0$ and $x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$, $\lambda_{1,2} < 0$. The mass slides towards the equilibrium. The point $(0, 0)$ in the (x, v) plane is a sink.
- Critical damping when $D = 0$ and $x(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t}$, $\lambda < 0$. The point mass slides towards the equillibrium after, possibly, one swing. The point $(0, 0)$ in the (x, v) plane is a sink,
- Damped vibration when $D < 0$ and $x(t) = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$. The mass oscillates around the equillibrium with decreasing amplitudes. The point $(0, 0)$ is a spiral sink.

Case 3: Forced vibration without damping

In addition to internal forces of the system there is an additional external force $f(t)$ acting on the system, so

$$\boxed{m\ddot{x} + kx = f(t)}.$$

The general solution is of the form

$$x(t, C_1, C_2) = C_1 \cos(\omega t) + C_2 \sin(\omega t) + x_p(t),$$

where x_p is a particular solution of the nonhomogeneous equations.

Example

Let $f(t) = a \sin \mu t$.

Using the method of intelligent guessing,

- if $\mu \neq \omega$, then $x_p(t) = A \sin \mu t + B \cos \mu t$
- if $\mu = \omega$, then $x_p = t(A \sin \omega t + B \cos \omega t)$, so the solutions of the equation are unbonded and incerase towards ∞ as $t \to \infty$ – the well known phenomenion of resonance occurs.

Case 4: Forced vibration with damping:

$$\boxed{m\ddot{x} + \beta\dot{x} + kx = f(t)}.$$

Example

Let $f(t) = a \sin \mu t$.
The general solution is of the form

$$x(t, C_1, C_2) = x_h + x_p = C_1 x_1(t) + C_2 x_2(t) + x_p(t)$$

where $x_p(t)$ is of the form $A \sin \mu t + B \cos \mu t$, and the two solutions x_1 and x_2 both converge to 0 as $t \to \infty$. For any C_1, C_2 the solution $x(t, C_1, C_2)$ asymptotically tends towards $x_p(t)$.