Mathematical modelling

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The dynamics of systems of 2 equations

For an autonomous linear system

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2, \quad \dot{x}_2 = a_{21}x_1 + a_{22}x_2,$$

the origin $(0,0)$ is always a stationary point, i.e., an equilibrium solution.

The eigenvalues of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

determine the type of the stationary point $(0,0)$ and the shape of the phase portrait.

We will assume that $\det A \neq 0$. Let λ_1, λ_2 be the eigenvalues of A. We also assume that there exist two linearly independent vectors v_1, v_2 of A (even if $\lambda_1 = \lambda_2$).

Case 1: $\lambda_1, \lambda_2 \in \mathbb{R}$

The general solution is

$$x(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2.$$

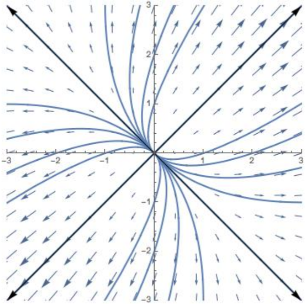
- If $C_1 = 0$, the trajectory $x_1(t)$ is a ray in the direction of v_2 if $C_2 > 0$, or $-v_1$ if $C_2 < 0$.
- Similarly, if $C_2 = 0$ the trajectory $x_2(t)$ is a ray in the direction of v_2 or $-v_2$.
- The behaviour of other trajectories depends on the signs of λ_1 and λ_2.

Subcase 1.1: $0 < \lambda_1 < \lambda_2$

- as $t \to \infty$, $x(t)$ asymptotically approaches the solution $\pm e^{\lambda_2 t} v_2$,
- as $t \to -\infty$, $x(t)$ asymptotically approaches the solution $\pm e^{\lambda_1 t} v_1$.

The point $(0,0)$ is a source.
Example. The general solution of the system $\dot{x}_1 = 3x_1 + x_2, \dot{x}_2 = x_1 + 3x_2$ is

$$x(t) = C_1 e^{4t} \begin{bmatrix} 1 & 1 \end{bmatrix}^T + C_2 e^{2t} \begin{bmatrix} -1 & 1 \end{bmatrix}^T.$$



Algorithm:

https://zalara.github.io/Algoritmi/phaseportrait_1_1.m

Subcase 1.2: $\lambda_2 < \lambda_1 < 0$

- as $t \to \infty$, $x(t)$ asymptotically approaches the solution $\pm e^{\lambda_1 t} v_2$,
- as $t \to -\infty$, $x(t)$ asymptotically approaches the solution $\pm e^{\lambda_2 t} v_1$.

The point $(0, 0)$ is a sink.

Example. The general solution of the system $\dot{x}_1 = -3x_1 - x_2, \dot{x}_2 = -x_1 - 3x_2$ is

$$x(t) = C_1 e^{-4t} \begin{bmatrix} 1 & 1 \end{bmatrix}^T + C_2 e^{-2t} \begin{bmatrix} -1 & 1 \end{bmatrix}^T.$$

Algorithm:

https://zalara.github.io/Algoritmi/phaseportrait_1_2.m

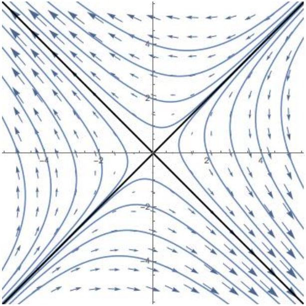
Subcase 1.3: $\lambda_1 < 0 < \lambda_2$

- as $t \to \infty$, $x(t)$ asymptotically approaches the solution $\pm e^{\lambda_2 t} v_2$,
- as $t \to -\infty$, $x(t)$ asymptotically approaches the solution $\pm e^{\lambda_1 t} v_1$.

The point $(0,0)$ is a saddle.

Example. The general solution of the system $\dot{x}_1 = x_1 - 3x_2, \dot{x}_2 = -3x_1 + x_2$ is

$$x(t) = C_1 e^{-2t} \begin{bmatrix} 1 & 1 \end{bmatrix}^T + C_2 e^{4t} \begin{bmatrix} 1 & -1 \end{bmatrix}^T.$$



Algorithm:

https://zalara.github.io/Algoritmi/phaseportrait_1_3.m

Subcase 2.1: $\lambda_{1,2} = \alpha \pm i\beta$, $\alpha \neq 0$

The general solution is

$$x(t) = e^{\alpha t}\left[(C_1 \cos(\beta t) + C_2 \sin(\beta t))u + (-C_1 \sin(\beta t) + C_2 \cos(\beta t))w\right].$$

Hence,

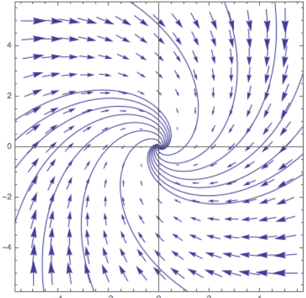
- if $\alpha < 0$, $x(t)$ spirals towards $(0,0)$ as $t \to \infty$, and
- if $\alpha > 0$, $x(t)$ spirals away from $(0,0)$ as $t \to \infty$.

The point $(0,0)$ is a spiral sink in the first case and a spiral source in the second case.

Example

$$\dot{x}_1 = -3x_1 + 2x_2, \dot{x}_2 = -x_1 - x_2$$

$$x(t) = e^{-2t} \cdot \left((C_1 \cos t + C_2 \sin t) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-C_1 \sin t + C_2 \cos t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$



Subcase 2.2: $\lambda_{1,2} = \pm i\beta$, $\alpha \neq 0$

The trajectories are periodic with period $2\pi/\beta$, i.e. the point $x(t)$ circles around $(0,0)$.

The point $(0,0)$ is a center.

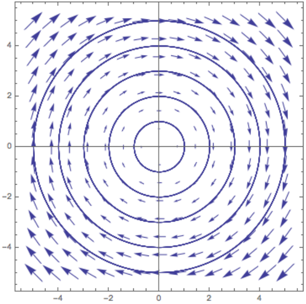
Example

$$\dot{x} = v, \quad \dot{v} = -\omega^2 x$$

$x(t) =$

$$(C_1 \cos(\omega t) + C_2 \sin(\omega t)) \begin{bmatrix} 1 \\ 0 \end{bmatrix} +$$

$$(-C_1 \sin(\omega t) + C_2 \cos(\omega t)) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Algorithm:

https://zalara.github.io/Algoritmi/phaseportrait_2_1.m
https://zalara.github.io/Algoritmi/phaseportrait_2_2.m

Nonlinear autonomous systems of equations

$$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2)$$

If $a = (a_1, a_2)$ is a critical point, that is,

$$f_1(a_1, a_2) = f_2(a_1, a_2) = 0,$$

then the behaviour of trajectories close to a is approximated by trajectories of the linearization of the system at the point a:

$$\dot{x}_1 \doteq \frac{\partial f_1}{\partial x_1}(x_1 - a_1) + \frac{\partial f_1}{\partial x_2}(x_2 - a_2), \quad \dot{x}_2 \doteq \frac{\partial f_2}{\partial x_1}(x_1 - a_1) + \frac{\partial f_2}{\partial x_2}(x_2 - a_2).$$

This is a linear homogeneous system with coefficient matrix the Jacobian matrix of the vector function $f(x)$:

$$\dot{x} \doteq Df(a)(x - a) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \end{bmatrix} (x - a).$$

The critical point is classified as a source, sink, saddle, spiral source, spiral sink or center depending on the eigenvalues of $Df(a)$.

In addition to critical points, that is, equillibrium solutions, a plane nonlinear system (that is, a nonlinear system of two differential equations) can also have limit cycles.

A limit cycle is a periodic solutions $x_\infty(t)$ such that for initial conditions $x(t_0) = x_0$ in a certain domain the corresponding solutions $x(t)$

- either asymptotically tend towards $x_\infty(t)$ as $t \to \infty$ – in this case x_∞ is an attracting limit cycle, or
- $x(t) \to x_\infty(t)$ as $t \to -\infty$ – in this case x_∞ is a repelling limit cycle.

Systems of more than two differential equations can exhibit much more complex, chaotic behaviour.

Algorithm:

https://zalara.github.io/Algoritmi/example_predator_prey_linearization.m

Differential equations of order 2

$$\ddot{x} = f(t, x, \dot{x})$$

The general solution is a two-parametric family

$$x = x(t, C_1, C_2).$$

A particular solution is given by specifying

- initial conditions: $x(t_0) = \alpha_0, \quad \dot{x}(t_0) = \alpha_1,$
  where the values of the solution and its derivative are given at some initial time t_0
  or
- boundary conditions: $x(a) = x_0, \quad x(b) = x_1$
  where values of the solution at different times a, b are given (i.e., on the boundary of some interval $[a, b]$)

Differential equations of order n

$$x^{(n)} = f(t, x, \dot{x}, \ldots, x^{(n-1)})$$

The general solution is an n-parametric family

$$x = x(t, C_1, \ldots, C_n).$$

A particular solution is given by

- initial conditions: $x(t_0) = \alpha_0, \ldots x^{(n-1)}(t_0) = \alpha_{n-1}$
  where the values of the solution and its first $(n-1)$ derivatives are given at some initial time t_0
  or
- boundary conditions
  where values of the solution or its derivatives are given in different times.

Linear DE's of order n

A linear DE (LDE) of degree n is of the form

$$x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = f(t). \tag{1}$$

The equation is

- homogeneous if $f(t) = 0$, and
- nonhomogeneous if $f(t) \neq 0$.
- The general solution of the homogeneous part is the family of all linear combinations
$$y(t) = C_1 x_1(t) + \cdots + C_n x_n(t)$$
of n linearly independent solutions $x_1(t), \ldots, x_n(t)$.
- If the coefficients $a_1(t), \ldots, a_n(t)$ are continuous functions, then for any $\alpha_0, \ldots, \alpha_n$ there exists exactly one solution satisfying the initial condition
$$x(t_0) = \alpha_0, \quad \dot{x}(t_0) = \alpha_1, \quad \ldots, \quad x^{(n-1)}(t_0) = \alpha_n.$$

LDEs with constant coefficients

Assume that the coefficient functions $a_1(t), \dots a_n(t)$ in a homogeneous LDE are constant:

$$x^{(n)} + a_{n-1}x^{(n-1)} \cdots + a_0 x = 0, \quad a_1, \dots a_n \in \mathbb{R} \tag{2}$$

Translating (2) to the system by the usual trick of introducing new variables

$$x_1 = x, \quad x_2 = x_1', \quad x_3 = x_2', \quad \cdots, \quad x_n = x_{n-1}',$$

(2) becomes

$$x_n' = -a_0 x_1 - a_1 x_2 - \dots - a_{n-1} x_n,$$

or matricially $\vec{x}' = A\vec{x}$:

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & \cdots & & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \vec{x}(t)$$

- The solutions to this system are of the form

$$x(t) = p_k(t)e^{\lambda t}v,$$

  where λ is the eigenvalue of A, $p_k(t)$ is a polynomial of degree k in t and v is the generalized eigenvector. (This follows most easily by the use of the Jordan form of the matrix.)
- In particular, if there are n linearly independent eigenvectors of the matrix A, then all polynomials p_k are constants and generalized eigenvectors are usual eigenvectors.
- By a simple calculation of expressing the determinant of $A - \lambda I$ according to the coefficients and cofactors of the last row, it turns out that the eigenvalues of A are precisely the roots of the characteristic polynomial corresponding to (2):

$$P(\lambda) := \lambda^n + a_{n-1}\lambda^{n-1} + \cdots a_1\lambda + a_0. \tag{3}$$

- A (trivial) fact with a nontrivial proof, called the fundamental theorem of algebra, states that a polynomial of degree n has exactly n roots, counted by multiplicity. In case the matrix A is real, these roots are real or complex conjugate pairs.

- From the roots of the characteristic polynomial (3), n linearly independent solutions of the LDE can be reconstructed.
- For every real root $\lambda \in \mathbb{R}$,

$$x(t) = e^{\lambda t}$$

  is a solution of the homogeneous LDE.
- For a complex conjugate pair of roots $\lambda = \alpha \pm i\beta$, the real and imaginary parts of the complex-valued exponential functions

$$e^{(\alpha \pm i\beta)t} = e^{\alpha t}(\cos(\beta t) + i \sin(\beta t))$$

  are two linearly independent solutions

$$x_1 = e^{\alpha t}\cos(\beta t), \quad x_2 = e^{\alpha t}\sin(\beta t).$$

Proposition

If a root (or a complex pair of roots) λ has multiplicity $k > 1$, then it can be shown that

$$e^{\lambda t}, te^{\lambda t}, \ldots t^{k-1}e^{\lambda t}$$

are all linearly independent solutions.

Proof of proposition

Let us prove the last fact by an interesting trick. We introduce the operator

$$L : \mathcal{C}^{(n)}(I) \to \mathcal{C}(I)$$

$$L(u) = u^{(n)} + a_{n-1}u^{(n-1)} \cdots + a_0 u,$$

where $\mathcal{C}^{(n)}(I)$ stands for the vector space of n-times continuously differentiable functions on the interval I and $\mathcal{C}(I)$ stands for the vector space of continuous functions on I.

Let λ_0 be the root of the characteristic polynomial (3) of multiplicity k, i.e.,

$$P(\lambda) = (\lambda - \lambda_0)^k Q(\lambda).$$

Let $0 \leq q \leq k$ by an integer. We will check that $t^q e^{\lambda t}$ solves (2).
Notice that

$$t^q e^{\lambda t} = \frac{d^q}{d\lambda^q} e^{\lambda t}.$$

For ease of notation we define $a_n := 1$. We have that:

$$L(t^q e^{\lambda t}) = \sum_{i=0}^{n} a_i \left(\frac{d^q}{d\lambda^q} e^{\lambda t} \right)^{(i)} = \sum_{i=0}^{n} a_i \frac{d^i}{dt^i} \left(\frac{d^q}{d\lambda^q} e^{\lambda t} \right)$$

$$= \frac{d^q}{d\lambda^q} \left(\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} e^{\lambda t} \right) = \frac{d^q}{d\lambda^q} \left(\sum_{i=0}^{n} a_i \lambda^i e^{\lambda t} \right) = \frac{d^q}{d\lambda^q} (P(\lambda) e^{\lambda t}).$$

Since

$$\frac{d^q}{d\lambda^q}(P(\lambda)e^{\lambda t}) = \sum_{i=1}^{q} \frac{d^i}{d\lambda^i}(P(\lambda)) \cdot Q_i(t, \lambda),$$

where $Q_i(t, \lambda)$ are functions of t, λ and

$$\frac{d^i}{d\lambda^i}(P(\lambda_0)) = 0, \quad \text{for} \quad i = 0, \ldots, q,$$

it follows that

$$L(t^q e^{\lambda_0 t}) = 0.$$