Mathematical modelling

Lecture 12, May 3rd, 2022

Faculty of Computer and Information Science
University of Ljubljana

2021/22

Transformating DEs of higher order into 1st order ODEs

The differential equation of order 2

$$\ddot{x} = f(t, x, \dot{x}) \tag{1}$$

can be transformed into a system of two order 1 DE's by introducing new variables:

$$\begin{aligned} x_1(t) &= x(t), \\ x_2(t) &= \dot{x}(t). \end{aligned}$$

Now DE (1) becomes

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= f(t, x_1(t), x_2(t)). \end{aligned}$$

An initial condition

$$x(t_0) = \alpha_0, \quad \dot{x}(t_0) = \alpha_1$$

is transformed into an initial condition

$$\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}.$$

In the same way a differential equation of order n

$$x^{(n)} = f(t, x, \dot{x}, \dots, x^{(n-1)})$$

can be transformed into a system of n differential equations of order 1 by introducing new dependent variables

$$\begin{aligned} x_1 &= x, \\ x_2 &= \dot{x}, \\ &\vdots \\ x_n &= x^{(n-1)}, \end{aligned} \tag{2}$$

and hence (2) becomes:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ f(t, x_1, x_2, \dots, x_n) \end{bmatrix}.$$

Example: We are given the differential equation of order 2

$$2\ddot{x} - 5\dot{x} + x = 0, \tag{3}$$

with initial conditions

$$x(3) = 6, \qquad \dot{x}(3) = -1. \tag{4}$$

We introduce new variables:

$$\begin{aligned} x_1(t) &= x(t), \\ x_2(t) &= \dot{x}(t), \end{aligned}$$

and hence (3) becomes the system

$$\begin{aligned} \dot{x_1}(t) &= x_2(t), \\ \dot{x_2}(t) &= \frac{5}{2}x_2 - \frac{1}{2}x_1. \end{aligned}$$

An initial conditions (4) becomes

$$x_1(3) = 6, \qquad x_2(3) = -1.$$

Numerical methods for a system of DEs

Numerical methods for a system of DEs work exactly in the same way as for a single equation, with the exception that the unknown function is a vector function

$$x(t) = \begin{bmatrix} x_1(t) & \cdots & x_n(t) \end{bmatrix}^T.$$

Given the system with initial condition

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1(t, x_1, \ldots, x_n) \\ \vdots \\ f_n(t, x_1, \ldots, x_n) \end{bmatrix}, \quad x(t_0) = x^{(0)} = \begin{bmatrix} x_1^{(0)} \\ \vdots \\ x_n^{(0)} \end{bmatrix},$$

we construct a recursive sequence of points

$$t_i = t_0 + ih, \quad x^{(i)} \doteq x(t_i), i \geq 0$$

where the vector $x^{(i)}$ is an approximation to the value of the exact solution $x(t_i)$, and h is the step size.

Euler's method and RK4

Euler's method:

$$t_{i+1} = t_i + h, \quad x^{(i+1)} = x^{(i)} + hf(t_i, x^{(i)}), \quad i \geq 0.$$

RK4 method:

$$t_{i+1} = t_i + h, \quad x^{(i+1)} = x^{(i)} + (k_1 + 2k_2 + 2k_3 + k_4)/6,$$

where

$$\begin{aligned}
k_1 &= hf(t_i, x^{(i)}), \\
k_2 &= hf(t_i + h/2, x^{(i)} + k_1/2), \\
k_3 &= hf(t_i + h/2, x^{(i)} + k_2/2), \\
k_4 &= hf(t_i + h, x^{(i)} + k_3).
\end{aligned}$$

Autonomous system of DE's - general case

A system of DEs is autonomous if the function $f : \mathbb{R}^n \to \mathbb{R}^n$ does not depend on t:

$$\dot{x} = f(x).$$

For an autonomous system, the tangent vector to a solution depends only on the point x and is independent of the time t at which the solution reaches a given point. In this case, the tangent vectors can be viewed as a directional field in the space $\mathbb{R}^n$.

In case of an autonomous system of 2 DE's:

$$f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{aligned} \dot{x} &= f_1(x, y), \\ \dot{y} &= f_2(x, y), \end{aligned}$$

gives a directional field in the (x, y) plane, which we call the phase plane of the system.

The general solution is a family of parametric curves or trajectories which respect the given directional field at every point (x, y).

The points where $f(x) = 0$ are stationary points or equilibrium points of the system.

At a stationary point $x_0 = x(t_0)$, $\dot{x}(t_0) = 0$, so $x(t) = x_0$ represents a constant, or equilibrium solution of the system.

Real life example of autonomous system

The predator-prey or Volterra-Lotka model is a famous system of DE's proposed by Alfred J. Lotka (1920) for modelling certain chemical reactions, and independently by Vito Volterra (1926) for dynamics of biological systems. It was later applied in economics and is used in a number of domains.

Two populations of species, for example rabbits and foxes, live together and depend on each other.

The number of rabbits (the prey) at time t is $R(t)$ and the number of foxes (the predators) is $F(t)$.

If they live apart, the rabbit, resp. fox, population grows, resp. declines, with the exponential law:

$$\dot{R} = kR, \quad k > 0, \quad \text{resp.} \quad \dot{F} = -rF, \quad r > 0.$$

If they live together, then interactions between rabbits and foxes cause a decline in the rabbit population and a growth of the fox population.
This (basic) predator-prey model is the following:

$$\dot{R} = kR - aRF, \quad \dot{F} = -rF + bFR, \quad a, b > 0.$$

The system has two stationary or equilibrium points:

$$kR - aRF = -rF + bFR = 0 \Rightarrow$$

$$\Rightarrow R = F = 0 \qquad \text{or} \qquad R = \frac{r}{b}, F = \frac{k}{a}.$$

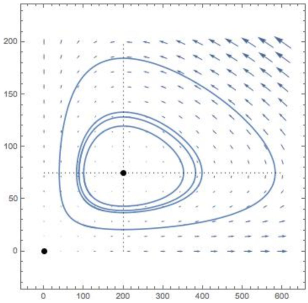
The meaning of these values is that the populations (ideally) coexist peacefully, with no fluctuations in the population sizes.

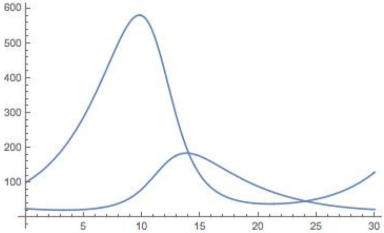
The left figure below shows the directional field and several solutions for the system

$$\dot{R} = 0.3R - 0.004RF \quad \dot{F} = -0.2F + 0.001FR$$

in the (R, F) plane.

The right figure shows dynamics of the population sizes $F(t)$ and $R(t)$ with respect to t:

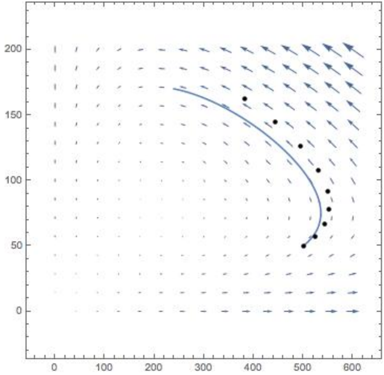




On the figure below, the blue curve is the exact solution and the black dots are approximations for function values for the system with initial condition

$$R(0) = 500, F(0) = 50$$

using Euler's method with step size $h = 0.5$:



Algorithm:

https://zalara.github.io/Algoritmi/example_predator_prey.m

The dynamics of systems of 2 equations

For an autonomous linear system

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2, \quad \dot{x}_2 = a_{21}x_1 + a_{22}x_2,$$

the origin $(0, 0)$ is always a stationary point, i.e., an equilibrium solution.

The eigenvalues of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

determine the type of the stationary point $(0, 0)$ and the shape of the phase portrait.

We will assume that $\det A \neq 0$. Let λ_1, λ_2 be the eigenvalues of A. We also assume that there exist two linearly independent vectors v_1, v_2 of A (even if $\lambda_1 = \lambda_2$).

Case 1: $\lambda_1, \lambda_2 \in \mathbb{R}$

The general solution is

$$x(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2.$$

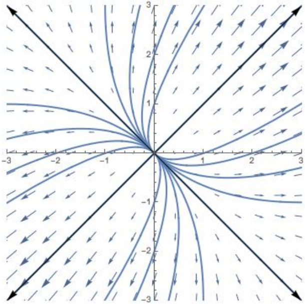
- If $C_1 = 0$, the trajectory $x_1(t)$ is a ray in the direction of v_2 if $C_2 > 0$, or $-v_1$ if $C_2 < 0$.
- Similarly, if $C_2 = 0$ the trajectory $x_2(t)$ is a ray in the direction of v_2 or $-v_2$.
- The behaviour of other trajectories depends on the signs of λ_1 and λ_2.

Subcase 1.1: $0 < \lambda_1 < \lambda_2$

- as $t \to \infty$, $x(t)$ asymptotically approaches the solution $\pm e^{\lambda_2 t} v_2$,
- as $t \to -\infty$, $x(t)$ asymptotically approaches the solution $\pm e^{\lambda_1 t} v_1$.

The point $(0, 0)$ is a source.
Example. The general solution of the system $\dot{x}_1 = 3x_1 + x_2, \dot{x}_2 = x_1 + 3x_2$ is

$$x(t) = C_1 e^{4t} \begin{bmatrix} 1 & 1 \end{bmatrix}^T + C_2 e^{2t} \begin{bmatrix} -1 & 1 \end{bmatrix}^T.$$



Algorithm:

https://zalara.github.io/Algoritmi/phaseportrait_1_1.m

Subcase 1.2: $\lambda_2 < \lambda_1 < 0$

- as $t \to \infty$, $x(t)$ asymptotically approaches the solution $\pm e^{\lambda_1 t} v_2$,
- as $t \to -\infty$, $x(t)$ asymptotically approaches the solution $\pm e^{\lambda_2 t} v_1$.

The point $(0,0)$ is a sink.

Example. The general solution of the system $\dot{x}_1 = -3x_1 - x_2, \dot{x}_2 = -x_1 - 3x_2$ is

$$x(t) = C_1 e^{-4t} \begin{bmatrix} 1 & 1 \end{bmatrix}^T + C_2 e^{-2t} \begin{bmatrix} -1 & 1 \end{bmatrix}^T.$$

Algorithm:

https://zalara.github.io/Algoritmi/phaseportrait_1_2.m

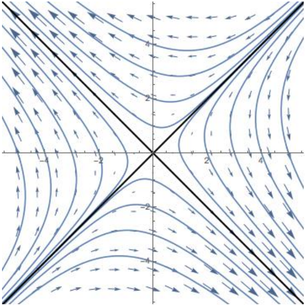
Subcase 1.3: $\lambda_1 < 0 < \lambda_2$

- as $t \to \infty$, $x(t)$ asymptotically approaches the solution $\pm e^{\lambda_2 t} v_2$,
- as $t \to -\infty$, $x(t)$ asymptotically approaches the solution $\pm e^{\lambda_1 t} v_1$.

The point $(0,0)$ is a saddle.

Example. The general solution of the system $\dot{x}_1 = x_1 - 3x_2, \dot{x}_2 = -3x_1 + x_2$ is

$$x(t) = C_1 e^{-2t} \begin{bmatrix} 1 & 1 \end{bmatrix}^T + C_2 e^{4t} \begin{bmatrix} 1 & -1 \end{bmatrix}^T.$$



Algorithm:

https://zalara.github.io/Algoritmi/phaseportrait_1_3.m