Mathematical modelling

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General solution of a linear DE

$$\boxed{y'(x) = f(x)y(x) + g(x)}. \tag{1}$$

1. The homogenous part is

$$y'(x) = f(x)y(x). \tag{2}$$

So the solution $y(x)$ to (2) is

$$\log|y| = \int \frac{dy}{y} = \int f(x)dx + C \;\Rightarrow\; y(x) = C \cdot e^{\int f(x)dx}$$

2. A particular solution of the nonhomogenous equation is obtained by the variation of the constant:

$$y(x) = C(x) \cdot e^{\int f(x)dx}. \tag{3}$$

$$y'(x) = C'(x) \cdot e^{\int f(x)dx} + C(x)f(x)e^{\int f(x)dx}. \tag{4}$$

Using that (1)=(4) and by inserting the RHS of (3) instead of $y(x)$ in (1), we obtain

$$C'(x) \cdot e^{\int f(x)dx} + C(x)f(x)e^{\int f(x)dx} = f(x)C(x) \cdot e^{\int f(x)dx} + g(x)$$

Hence

$$C'(x) \cdot e^{\int f(x)dx} = g(x),$$

and so

$$C(x) = \int (g(x) e^{-\int f(x)dx}) dx.$$

Proposition

The solution of (1) *is*

$$y(x) = e^{\int f(x)dx} (C + \int (g(x) e^{-\int f(x)dx}) dx).$$

In the example $t^2 \dot{x} + tx = 1$ (or $\dot{x} = -\frac{1}{t} x + \frac{1}{t^2}$) above we get

$$\begin{aligned} x(t) &= e^{\int -\frac{1}{t} dt} \left(C + \int \left(\frac{1}{t^2} e^{\int \frac{1}{t} dt} \right) dt \right) \\ &= e^{\log |\frac{1}{t}|} \left(C + \int \left(\frac{1}{t^2} t \right) dt \right) \\ &= \frac{1}{t} (C + \log |t|). \end{aligned}$$

Real life example: Newton's second law

A ball of mass m kg is thrown vertically into the air with initial velocity $v_0 = 10$ m/s. We follow its trajectory. By Newton's second law of motion,

$$F = ma,$$

where m is the mass, $a = \dot{v} = \ddot{x}$ is acceleration and v velocity, and F is the sum of forces acting on the ball.

- Assuming no air friction the model is

  $$m\dot{v} = -mg,$$

  where g is the gravitational constant. The solution is

  $$v = -gt + C \quad \text{where } C \text{ is a constant.}$$

- Assuming the linear law of resistance (drag) $F_u = -kv$ the model is

  $$m\dot{v} = -mg - kv.$$

  The solution is $v = v_h + v_p$ where

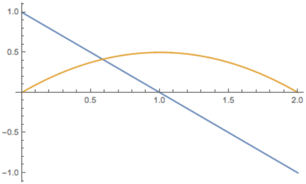
  $$v_h = Ce^{-kt/m} \quad \text{and} \quad v_p = -mg/k.$$

Motion of ball in the case $m = 1$, $k = 1$ and approximating $g \doteq 10$ (we will omit units)

Model | Velocity and position | Solution

$ma = -mg$

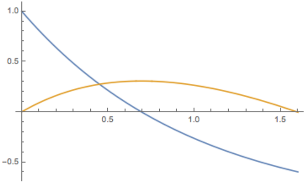
$\dot{v} = -10$



$v(t) = -10t + 10$
$x(t) = -5t^2 + 10t$

$ma = -mg - kv$

$\dot{v} = -v - 10$



$v(t) = 20e^{-t} - 10$
$x(t) = 20 - 20e^{-t} - 10t$

The ball reaches the top at time t where $v(t) = 0$ and the ground at time t where $x(t) = 0$.

- Assuming no friction, the ball is at the top at $t = 10$.

  At time $t = 1$, $x(t) = 0$, so it takes the same time going up and falling down.

- Assuming linear friction, the ball reaches the top at $t = \log 2$.

  At time $2 \log 2$, $x(2 \log 2) = 20 - 5 - 20 \log 2 > 0$ so it takes longer falling down than going up.

Homogeneous DE

A homogeneous (nonlinear) DE is of the form

$$\dot{x} = f\left(\frac{x}{t}\right). \tag{5}$$

The solution is obtained by introducing a new dependent variable

$$u = \frac{x}{t}.$$

Hence $x = ut$ and differentiating with respect to t we get

$$\dot{x} = \dot{u}t + u. \tag{6}$$

Plugging (6) into (5) we get

$$\dot{u}t + u = f(u). \tag{7}$$

Rearranging (7) we obtain

$$t\dot{u} = f(u) - u,$$

which is a separable DE.

Example (Homogeneous DE)

$$\boxed{y' = \frac{y - x}{x}}$$

can be written as

$$y' = \frac{y}{x} - 1. \tag{8}$$

Introducing a new dependent variable

$$u = \frac{y}{x},$$

plugging in (8), we get

$$u'x + u = u - 1. \tag{9}$$

This is equivalent to

$$u'x = -1$$

and hence

$$u = \frac{y}{x} = \log\left(\frac{C}{x}\right).$$

Orthogonal trajectories

Given a 1-parametric family of curves

$$F(x, y, a) = 0 \quad \text{where} \quad a \in \mathbb{R},$$

an orthogonal trajectory is a curve

$$G(x, y) = 0$$

that intersects each curve from the given family at a right angle.

Algorithm to obtain orthogonal trajectories:

1. The family $F(x, y, a) = 0$ is the general solution of a 1st order DE, that is obtained by differentiating the equation with respect to the independent variable (using implicit differentiation) and eliminating the parameter a.
2. By substituting y' for $-1/y'$ in the DE for the original family, we obtain a DE for curves with orthogonal tangents at every point of intersection.
3. The general solution to this equation is the family of orthogonal trajectories to the original equation.

Example (Orthogonal trajectories to the family of circles)

Let us find the orthogonal trajectories to the family of circles through the origin with centers on the y axis:

$$x^2 + y^2 - 2ay = 0. \tag{10}$$

Differentiating (10) w.r.t. the independent variable gives

$$2x + 2yy' - 2ay' = 0. \tag{11}$$

Expressing a from (11) gives

$$a = \frac{x}{y'} + y. \tag{12}$$

Inserting (12) into (10) we obtain the DE for the given family

$$x^2 - y^2 - \frac{2xy}{y'} = 0. \tag{13}$$

Next we express y' from (13) and obtain

$$y' = \frac{2xy}{x^2 - y^2}. \tag{14}$$

The DE for orthogonal trajectories is obtained by substituting y' for $-1/y'$ in (14) to obtain

$$-\frac{1}{y'} = \frac{2xy}{x^2 - y^2}, \tag{15}$$

which is equivalent to

$$y' = -\frac{x^2 - y^2}{2xy}. \tag{16}$$

(16) is a homogeneous DE:

$$y' = -\frac{x^2 - y^2}{2xy} = -\frac{x}{2y} + \frac{y}{2x}$$

By introducing $y = ux$ we obtain

$$u'x + u = -\frac{1}{2u} + \frac{u}{2} \quad \Rightarrow \quad u'x = -\frac{1 + u^2}{2u} \quad \Rightarrow \quad \frac{2udu}{1 + u^2} = -\frac{dx}{x}$$

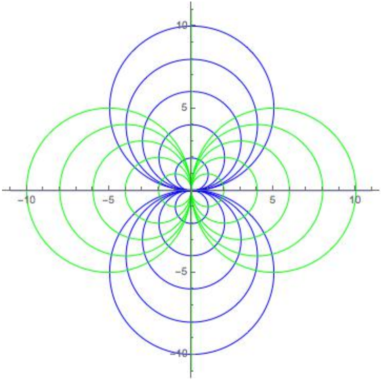
$$\Rightarrow \quad \log(1 + u^2) = -\log x + \log C \quad \Rightarrow \quad 1 + u^2 = \frac{C}{x},$$

Plugging in $u = \frac{y}{x}$ again gives the general solution

$$x^2 + y^2 = Cx.$$

Orthogonal trajectories to circles through the origin with centers on the y axis are circles through the origin with centers on the x axis.

Both families together form an orthogonal net:



Exact ODEs

Notice first that a 1st order DE

$$\dot{x} = f(t, x)$$

can be rewritten in the form

$$M(t, x)dt + N(t, x)dx = 0. \tag{17}$$

Recall that the differential of a function $u(t, x)$ is equal to

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx = \left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right) \cdot (dt, dx),$$

where $\cdot$ denotes the usual inner product in $\mathbb{R}^2$.

DE (17) is exact if there exists a differentiable function $u(t, x)$ such that

$$\frac{\partial u}{\partial t} = M(t, x) \quad \text{and} \quad \frac{\partial u}{\partial x} = N(t, x).$$

Proposition

If the DE (17) *is exact, then the solutions are level curves of the function* u*:*

$$u(t,x) = C, \qquad \textit{where} \quad C \in \mathbb{R}.$$

Recall from Calculus that if u has continuous second order partial derivatives then

$$\frac{\partial u}{\partial x \partial t} = \frac{\partial u}{\partial t \partial x}.$$

Proposition

The necessary condition for the DE (17) *to be exact is*

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}. \tag{18}$$

Moreover, if M *and* N *are differentiable for every* $(t,x) \in \mathbb{R}^2$*, the condition* (18) *is also sufficient.*

A potential function u can be determined from the following equality

$$u(x,t) = \int M(t,x)\,dt + C(x) = \int N(t,x)\,dx + D(t),$$

where $C(x)$ and $D(t)$ are some functions.

Example. The DE

$$x + ye^{2xy} + xe^{2xy}y' = 0$$

can be rewritten as

$$(x + ye^{2xy})dx + xe^{2xy}dy = 0.$$

The equation is exact since

$$\frac{\partial(x + ye^{2xy})}{\partial y} = \frac{\partial(xe^{2xy})}{\partial x} = (e^{2xy} + 2xye^{2xy}).$$

A potential function is equal to

$$\begin{aligned} u(x,y) &= \int (x + ye^{2xy})\,dx = \frac{x^2}{2} + \frac{1}{2}e^{2xy} + C(y) \\ &= \int (xe^{2xy})\,dy = \frac{1}{2}e^{2xy} + D(x), \end{aligned}$$

Defining $C(y) = 0$ and $D(x) = x^2/2$, we get $u(x,y) = \frac{x^2}{2} + \frac{1}{2}e^{2xy}$. The general solution is the family of level curves $u(x,y) = E$, where $E \in \mathbb{R}$.

Geometric picture of ODEs

Let $D \subset \mathbb{R}^2$ be the domain of the function $f(x, y)$. For each point $(x, y) \in D$ the DE

$$y' = f(x, y)$$

gives the value y' of the coefficient of the tangent to the solution $y(x)$ through this specific point, that is, the direction in which the solution passes through the point.

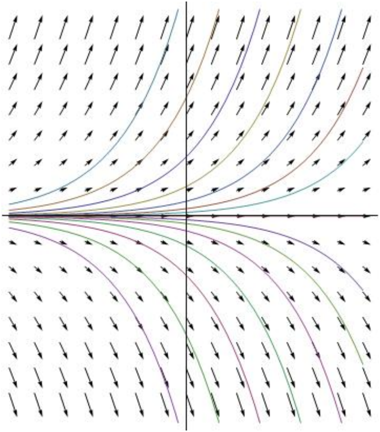
All these directions together form the directional field of the equation.

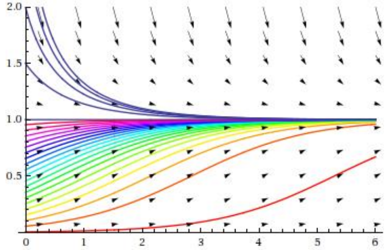
A solution of the equation is represented by a curve $y = y(x)$ that follows the given directions at every point x, i.e., the coefficient of the tangent corresponds to the value $f(x, y(x))$.

The general solution to the equation is a family of curves, such that each of them follows the given direction.

Directional fields and solutions of

$$y' = ky \qquad\qquad y' = ky(1-y)$$





Examples: https://zalara.github.io/Algoritmi/example_direction_fields.m

Theorem (Existence and uniqueness of solutions)

If $f(x, y)$ is continuous and differentiable with respect to y on the rectangle

$$D = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b], \quad a, b > 0$$

then the DE with initial condition

$$y' = f(x, y), \quad y(x_0) = y_0,$$

has a unique solution $y(x)$ defined at least on the interval

$$[x_0 - \alpha, x_0 + \alpha], \quad \alpha = \min\left\{a, \frac{b}{M}, \frac{1}{N}\right\},$$

where

$$M = \max\{f(x, y) \colon (x, y) \in D\} \text{ and } N = \max\left\{\frac{\partial f(x, y)}{\partial y} \colon (x, y) \in D\right\}.$$

Proof: https://en.wikipedia.org/wiki/Picard%E2%80%93Lindel%C3%B6f_theorem.

Numerical methods for solving DE's

We are given the DE with the initial condition

$$y'(x) = f(y, x), \quad y(x_0) = y_0.$$

Instead of analytically finding the solution $y(x)$, we construct a recursive sequence of points

$$x_i = x_0 + ih, \quad y_i \doteq y(x_i), \quad i \geq 0$$

where y_i is an approximation to the value of the exact solution $y(x_i)$, and h is the step size.

A number of numerical methods exists, the choice depends on the type of equation, desired accuracy, computational time,...

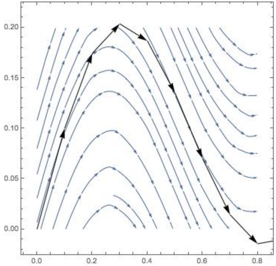
We will first look at the simplest and best known method and then a more practical improvement.

Euler's method

Euler's method is the simplest and most intuitive approach to numerically solve a DE.

At each step the value y_{i+1} is obtained as the point on the tangent to the solution through (x_i, y_i) at $x_{i+1} = x_i + h$:

- initial condition: (x_0, y_0)
- for each i: $x_{i+1} = x_i + h$, $y_{i+1} = y_i + hf(x_i, y_i)$.



The point (x_{i+1}, y_{i+1}) typically lies on a different particular solution than (x_i, y_i), at each step, the error at each step is of order $\mathcal{O}(h^2)$. The cumulative error is of order $\mathcal{O}(h)$.