Mathematical modelling

Lecture 9, April 12th, 2022

Faculty of Computer and Information Science
University of Ljubljana

2021/22

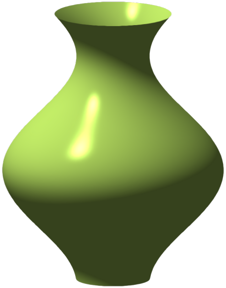
Surfaces of revolution

A surface of revolution is obtained by revolving a curve $x = x(u), z = z(u)$ in the (x, z)-plane around the z axis:

$$f(u, v) = \begin{bmatrix} x(u)\cos v \\ x(u)\sin v \\ z(u) \end{bmatrix}$$

$u \in [a, b]$

$v \in [0, 2\pi]$,



$x = 2 + \cos t, z = t$, from wikipedia

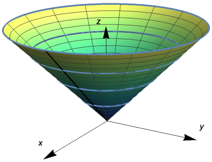
Coordinate curves:

- $u = u_0$, horizonal circle $\begin{bmatrix} x(u_0)\cos v \\ x(u_0)\sin v \\ z(u_0) \end{bmatrix}$,
- $v = v_0$, original curve rotated by the angle v_0

Example

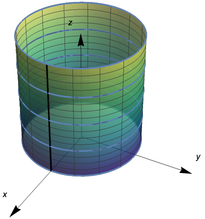
Revolving the line $z = x = u$: a cone

$$\begin{bmatrix} u\cos v \\ u\sin v \\ u \end{bmatrix}$$



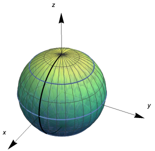
Revolving the line $x = a$: a cylinder

$$f(u, v) = \begin{bmatrix} a\cos v \\ a\sin v \\ u \end{bmatrix}$$

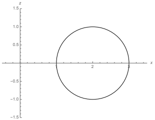


Revolving the half-circle
$x = \cos t, z = \sin t, -\pi/2 \leq t \leq \pi/2$: a sphere

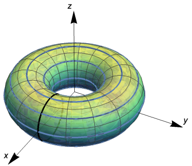
$$f(t, v) = \begin{bmatrix} \cos t \cos v \\ \cos t \sin v \\ \sin t \end{bmatrix}$$



Revolving the circle $x = 2 + \cos t, z = \sin t$, $0 \leq t \leq 2\pi$: a torus



$$f(u, v) = \begin{bmatrix} (2 + \cos t) \cos v \\ (2 + \cos t) \sin v \\ \sin t \end{bmatrix}$$



Matlab file: `https://zalara.github.io/Algoritmi/surface_of_revolution.m`

Smooth surfaces

Let $\mathbf{r}_0 = f(u_0, v_0)$ be a point on the surface.

Coordinate curves through this point:

- $f(u_0, v)$ with parameter v and tangent vector $f_v(u_0, v_0)$,
- $f(u, v_0)$ with parameter u and tangent vector $f_u(u_0, v_0)$.

The parametric surface is smooth at the point $f(u_0, v_0)$, if both tangent vectors exist and

$$f_u(u_0, v_0) \times f_v(u_0, v_0) \neq 0.$$

The vector $\mathbf{n}_0 = f_u(u_0, v_0) \times f_v(u_0, v_0)$ is the normal vector to the surface at the point $\mathbf{r}_0$.

Tangent plane

If the surface is smooth at a point $\mathbf{r}_0 = f(u_0, v_0)$ then it has a tangent plane at this point that is given:

- in implicit form by $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n}_0 = 0$
- in parametric form by

$$\mathbf{r}(u, v) = \mathbf{r}_0 + u f_u(u_0, v_0) + v f_v(u_0, v_0) = L_{(u_0, v_0)}(u, v)$$

where $L_{(u_0,v_0)}(u, v) = f(u_0, v_0) + Df(u_0, v_0) \begin{bmatrix} u \\ v \end{bmatrix}$
is the linear approximation and

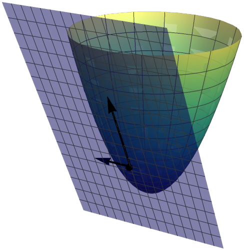
$$Df(u_0, v_0) = \begin{bmatrix} x_u(u_0, v_0) & x_v(u_0, v_0) \\ y_u(u_0, v_0) & y_v(u_0, v_0) \\ z_u(u_0, v_0) & z_v(u_0, v_0) \end{bmatrix}$$

is the Jacobian.

Problem: find the tangent plane to the surface $f(u, v) = \begin{bmatrix} u\cos v \\ u\sin v \\ u^2 \end{bmatrix}$ at $u = 1, v = \pi/2$.

Since $f_u(u, v) = \begin{bmatrix} \cos v \\ \sin v \\ 2u \end{bmatrix}$ and $f_v(u, v) = \begin{bmatrix} -u\sin v \\ u\cos v \\ 0 \end{bmatrix}$ the tangent plane in parametric form is

$$\mathbf{r}(u, v) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + u \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + v \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$



In implicit form:

$$\mathbf{n} = f_u(u_0, v_0) \times f_v(u_0, v_0) = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \times \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

so:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = -2(y - 1) + (z - 1) = 0.$$

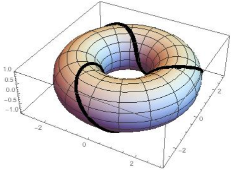
A curve $\alpha(t) = \begin{bmatrix} u(t) \\ v(t)) \end{bmatrix}$ in the (u, v)-plane corresponds to a curve on the parametric surface:

$$f(\alpha(t)) = \begin{bmatrix} x(u(t), v(t)) \\ y(u(t), v(t)) \\ z(u(t), v(t)) \end{bmatrix}$$

Example

The line $\alpha(t) = \begin{bmatrix} 3t \\ t \end{bmatrix}$ corresponds to a curve on the torus

$$(f \circ \alpha)(t) = \begin{bmatrix} (2 + \sin 3t) \cos t \\ (2 + \sin 3t) \sin t \\ \cos 3t \end{bmatrix}$$



An application: the configuration space of a robot

A robot, or a mechanical device, is described by its

- work space : the space of points reached by the end effector
- configuration space: the space of parameter values that determine the position of the robot

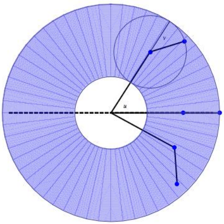
The number of parameters is the degrees of freedom, (DOF), this determines the dimension of the configuration space.

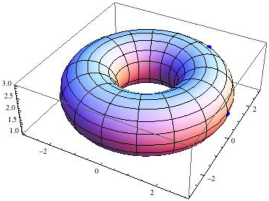
If DOF=2, thet the configuration space is (often) a parametric surface.

Classical example: A robotic arm with two links of lengths l_1 and l_2, $l_2 < l_1$ and two rotational joints.

The work space is a ring with interior circle of radius $l_1 - l_2$ and exterior circle of radius $l_1 + l_2$,

The configuration space is parametrized by the angles u and v in the joints, the two independent rotations can be represented by a torus: $\begin{bmatrix} (a + b\cos u)\cos v \\ (a + b\cos u)\sin v \\ b\sin u \end{bmatrix}$, $b < a, u \in [0, 2\pi], v \in [0, 2\pi]$





In robot motion planning, the motion of the robotic arm from point T_0 to T_1 in the work space is directed by a curve, or path, in the configuration space.

Chapter 5:

Differential equations and dynamic models

- Ordinary differential equation (ODE)
  - Definition and examples
  - Solving first order ODEs
    - Separable ODEs
    - First order linear ODEs
    - Homogeneous ODEs
  - Orthogonal trajectories
  - Exact ODEs
  - Geometric picture of ODEs
- Systems of first order ODEs
- Numerical methods for solving ODEs
- Autonomous system of ODEs
- Dynamics of systems of 2 linear ODEs
- Linear ODEs of order n
- Application - vibrating systems

Differential equations and dynamic models

Ordinary differential equation, ODE, is an equation of an unknown function and an independent variable. ODE relates the independent variable with the function and its derivatives.

If t is an independent variable, $x(t)$ is a function of t, then the ODE is of the form:

$$F(t, x, \dot{x}, \ddot{x}, \dots, x^{(n)}) = 0.$$

Similarly if x is an independent variable, $y(x)$ a function of x, then the ODE is of the form:

$$F(x, y, y', y'', \dots, y^{(n)}) = 0.$$

The order of a differential equation is the order of the highest derivative.

Examples of ODEs

- $\dot{x} - 3t^2 = 0.$

  So,
  $$\frac{dx}{dt} = 3t^2 \;\Rightarrow\; x(t) = t^3 + C, \quad \text{where } C \text{ is a constant.}$$
  If we want to determine C, we need an additional condition, e.g., initial condition $x(0) = x_0$, $x_0 \in \mathbb{R}$, or any other condition $x(t_0) = x_0$, $x_0 \in \mathbb{R}$.

- $y''(x) + 2y'(x) = 3y(x).$

  We will learn how to solve such an ODE, but right now let us only check that $y(x) = Ce^{-3x}$, $C \in \mathbb{R}$ a constant, is a solution:
  - Calculate $y''(x)$, $y'(x)$:
    $$y'(x) = -3Ce^{-3x}, \quad y''(x) = 9Ce^{-3x}.$$
  - Plug into the given ODE:
    $$9Ce^{-3x} - 6Ce^{-3x} = 3Ce^{-3x}.$$

- $\cos t \cdot \ddot{x} - 3t^4 \cdot \dot{x} + 5e^t = 0.$

  Such ODE's cannot be solved analytically (or are at least hard to solve). We will learn how to solve such ODE's by using numerical methods.

Partial differential equation, PDE, is an equation for an unknown function u of $n \geq 2$ independent variables, e.g., for $n = 2$ we have

$$F(x, y, u_x, u_y, u_{xx}, \ldots) = 0,$$

where x, y are the independent variables.

We will not consider PDE's, from now on DE means an ODE.

Applications of DEs

Differential equations are used for modelling a deterministic process: a law relating a certain quantity depending on some independent variable (for example time) with its rate of change, and higher derivatives.

1. Newton's law of cooling:

$$\dot{T} = k(T - T_\infty), \tag{1}$$

   where $T(t)$ is the temperature of a homogeneous body (can of beer) at time t, T_0 is the initial temperature at time $t_0 = 0$, T_∞ is the temperature of the environment, k is a constant (heat transfer coefficient).

   (1) is an example of a separable ODE and also the first order linear ODE. We will see shortly how to solve such types of ODE's. For now you can check easily by yourself that the solution is

$$T(t) = (T_0 - T_\infty)e^{kt}.$$

2. Radioactive decay:

$$\dot{y}(t) = -ky(t), \quad k = \frac{\log 2}{t_{1/2}},$$

where $y(t)$ is the remaining quantity of a radioactive isotope at time t, $t_{1/2}$ is the half-life and k is the decay constant. The solution is

$$y(t) = Ce^{-kt}, \quad \text{where } C \text{ is a constant.}$$

Let's verify, that $t_{1/2}$ really represents the time in which the amount of the isotope decreases to half of its current amount. At time $t = 0$ the amount is $y(0) = Ce^0 = C$. We have to check that $y(t_{1/2}) = \frac{C}{2}$:

$$y(t_{1/2}) = Ce^{-\frac{k \log 2}{k}} = Ce^{-\log 2} = Ce^{\log 1/2} = \frac{C}{2}.$$

3. Simple harmonic oscillator:

$$\ddot{x} + \omega x = 0.$$

Solution of a DE

The function $x(t)$ is a solution of a DE

$$F(t, x, \dot{x}, \ddot{x}, \ldots, x^{(n)}) = 0$$

on an interval I if it is at least n times differentiable and satisfies the identity

$$F(t, x(t), \dot{x}(t), \ddot{x}(t), \ldots, x^{(n)}(t)) = 0$$

for all $t \in I$.

Analytically solving a DE is typically very difficult, very often impossible.

To find approximate solutions we use different simplifications and numerical methods.

First order ODEs

We will (mostly) consider first order ODEs in the form

$$\dot{x} = f(t, x).$$

- The general solution is a one-parametric family of solutions $x = x(t, C)$.
- A particular solution is a specific function from the general solution, that usually satisfies some initial condition $x(t_0) = x_0$.
- A singular solution is an exceptional solution that is not part of the general solution.

We will first look at some simple types of 1.-st order DEs that are analytically solvable.

Separable DE

A separable DE is of the form

$$\dot{x} = f(t)g(x). \tag{2}$$

This can be solved by:

- Inserting $\dot{x} = \dfrac{dx}{dt}$ into (2):

$$\frac{dx}{dt} = f(t)g(x). \tag{3}$$

- Separating variables in (3):

$$\frac{dx}{g(x)} = f(t)\,dt. \tag{4}$$

- Integrating both sides of (3):

$$\int \frac{1}{g(x)}\,dx = \int f(t)\,dt + C$$

Example 1 of a separable DE

$$\boxed{\dot{x} = kx} \quad \text{where } k \in \mathbb{R} \text{ is a fixed real number} \tag{5}$$

- $$\frac{dx}{dt} = kx,$$
- $$\frac{dx}{x} = kdt,$$
- $$\log|x| = \int \frac{dx}{x} = \int k\,dt = kt + C,$$

where C is a constant and so

$$|x| = e^{kt+C}$$

is a general solution to (5). Clearly, $x(t) = 0$ is also a solution of the equation. By introducing a new constant e^C which, by abuse of notation, we again denote by C, this is equivalent to

$$x(t) = Ce^{kt}, C \in \mathbb{R}.$$

Example 2 of a separable DE

$$\boxed{\dot{x} = kx(1-x)} \quad \text{where } k \in \mathbb{R} \text{ is a fixed real number} \tag{6}$$

- $$\frac{dx}{dt} = kx(1-x),$$
- $$\frac{dx}{x(1-x)} = kdt,$$
- By the method of partial fractions we get

$$\log\left|\frac{x}{1-x}\right| = \log|x| - \log|1-x| = \int \frac{dx}{x} - \int \frac{dx}{1-x} = \int k\,dt = kt + C,$$

where C is a constant and so

$$\frac{x}{1-x} = Ce^{kt}.$$

Expressing $x(t)$ we get

$$x(t) = \frac{1}{Ce^{-kt}+1} \tag{7}$$

is a general solution to (6). $x(t)$ from (7) is called a logistic function.

Example 3 of a separable DE

$$\boxed{y' = \frac{-x}{ye^{x^2}},} \quad y(0) = 1. \tag{8}$$

- $$\frac{dy}{dx} = \frac{-x}{ye^{x^2}},$$
- $$ydy = -xe^{-x^2}dx,$$
- Integrating:
$$\frac{y^2}{2} = \int ydy = \int (-xe^{-x^2})dx = \frac{1}{2}e^{-x^2} + C,$$
where C is a constant.
- $\frac{1}{2} = \frac{y^2(0)}{2} = \frac{1}{2} + C \;\Rightarrow\; C = 0.$

Expressing $y(x)$ we get $y(x) = \pm\sqrt{e^{-x^2}}$ and since $y(0) > 0$ we have

$$y(x) = \sqrt{e^{-x^2}}.$$

Real life DE example: population growth

Let $x(t)$ be the size of a population (bacteria, trees, people, ...) at time t. The most common models for population growth are:

- exponential growth: the growth rate is proportional to the size, modelled by $\dot{x} = kx$, with the solution the exponential function $x(t) = x_0 e^{kt}$, where $x_0 = x(0)$ is the initial population size.
- logistic growth: the growth rate is proportional to the size and the resources, modelled by $\dot{x} = kx(1 - x/x_{max})$, where x_{max} is the capacity of the environment, i.e., maximal population size that it still supports, with the solution is the logistic function.
- general model: the growth rate is proportional to the size, but the proportionality factor depends on time and size, modelled by $\dot{x} = k(x,t)f(x)$; the equation is not separable and is analytically solvable only in very specific cases.

Real life DE example: information spreading

$x(t)$ is the ratio of people in a given group that at time t knows a certain piece of information.

Let $x_0 = x(t_0)$ be the 'informed' ratio at time $t = t_0$.

Consider two possible models:

- spreading through an external source: the rate of change is proportional to the uninformed ratio $\dot{x} = k(1 - x)$ with $x_0 = 0$,
- spreading through "word of mouth" the rate of change is proportional to the number of encounters between informed and uninformed members $\dot{x} = kx(1 - x)$ logistic law, again, with $x_0 > 0$.

First order linear ODE

A first order linear DE is of the form

$$\dot{x} + f(t)x = g(t) \tag{9}$$

The equation is homogeneous if $g(t) = 0$ and nonhomogenous if $g(t) \neq 0$.

A homogeneous part of (9),

$$\dot{x} + f(t)x = 0, \tag{10}$$

has a general solution of the form

$$Cx_h(t), \tag{11}$$

where $C \in \mathbb{R}$ is a constant and $x_h(t)$ is a particular solution. Indeed:

- Every $x(t)$ of the form (11) is a solution of (10):

$$\begin{aligned} x'(t) + f(t)x(t) &= (Cx_h)'(t) + f(t)Cx_h(t) \\ &= Cx_h'(t) + f(t)Cx_h(t) \\ &= C(x_h'(t) + f(t)x_h(t)) \\ &= 0 \end{aligned}$$

- If $x(t)$ is a solution of (10), then it must be of the form (11). Indeed, since $x(t)$ and $x_h(t)$ both solve (10),

$$\begin{aligned}\left(\frac{x(t)}{x_h(t)}\right)' &= \frac{x'(t)x_h(t) - x(t)x_h'(t)}{x_h^2(t)} \\ &= \frac{-f(t)x(t)x_h(t) + f(t)x(t)x_h(t)}{x_h^2(t)} \\ &= 0.\end{aligned}$$

  Hence, $\frac{x(t)}{x_h(t)} = C$ for some constant C and $x(t)$ is of the form (11).

Let $x_p(t)$ be any particular solution of (9):

$$x_p'(t) + f(t)x_p(t) = g(t). \tag{12}$$

The general solution of (9) is a sum

$$x(t) = Cx_h(t) + x_p(t). \tag{13}$$

Indeed:

- Every $x(t)$ of the form (13) is a solution of (9):

$$\begin{aligned} x'(t) + f(t)x(t) &= (Cx_h(t) + x_p(t))' + f(t)(Cx_h(t) + x_p(t)) \\ &= Cx_h'(t) + x_p'(t) + f(t)Cx_h(t) + f(t)x_p(t) \\ &= (Cx_h'(t) + f(t)Cx_h(t)) + (x_p'(t) + f(t)x_p(t)) \\ &= 0 + g(t), \end{aligned}$$

  where we used (12) in the last equality.

- If $x(t)$ is a solution of (9), then it must be of the form (13). Indeed, since $x(t)$ and $x_p(t)$ both solve (9), $x(t) - x_p(t)$ solves the homogenous part (10) of (9). Hence, $x(t) - x_p(t) = Cx_h(t)$ for some C and $x(t) = Cx_h(t) + x_p(t)$.

The particular solution x_p can be obtained by variation of the constant, that is, by substituting the constant C is the homogenous solution by an unknown function $C(t)$ which is then determined from the equation.

Example of a linear ODEs

$$\boxed{t^2\dot{x} + tx = 1}, \quad \boxed{x(1) = 2}. \tag{14}$$

1. The homogenous part is

$$t^2\dot{x} + tx = 0. \tag{15}$$

   So the solution x_h to (15) is

$$t^2\,dx = -tx\,dt \;\Rightarrow\; \frac{dx}{x} = -\frac{dt}{t} \;\Rightarrow\; \log|x| = -\log|t| + \log C = \log\frac{C}{|t|}$$

$$\Rightarrow\; x_h = \frac{C}{t}.$$

2. A particular solution of the nonhomogenous equation is obtained by variation of the constant:

$$x = \frac{C(t)}{t}, \quad \dot{x} = \frac{C'(t)t - C(t)}{t^2}$$

   by inserting into (14) we obtain

$$C'(t)t - C(t) + C(t) = 1 \;\Rightarrow\; C'(t) = \frac{1}{t} \;\Rightarrow\; C(t) = \log|t|.$$

3. So the general solution of the nonhomogenous equation is

$$x(t) = \frac{C}{t} + \frac{\log|t|}{t}. \tag{16}$$

4. Finally, since $x(1) = 2$, we get by plugging $t = 1$ into (16)

$$2 = x(1) = C$$

   and hence the solution of (14) is

$$x(t) = \frac{2 + \log|t|}{t}.$$