Mathematical modelling

Lecture 8, April 5th, 2022

Faculty of Computer and Information Science
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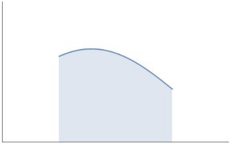
Areas bounded by plane curve

I. Let $f(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad t \in [a, b]$

$x'(t) > 0$



The area of the quadrilateral bounded by the curve and the x-axis is

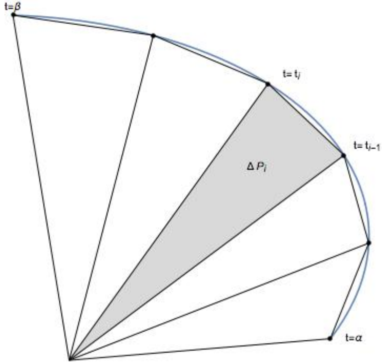
$$P = \int_{x(a)}^{x(b)} |y(x)|\, dx = \int_a^b |y(t)| x'(t)\, dt$$

Problem: the area under one arc of the cycloid:

$$x(t) = at - a\sin t, \quad y(t) = a - a\cos t,$$

$$P = \int_0^{2\pi} a^2 (1 - \cos t)^2\, dt = a^2 \int_0^{2\pi} \left(\frac{3}{2} - 2\cos t + \frac{1}{2}\cos(2t) \right) dt = 3a^2\pi.$$

II. The area of the triangular region bounded by the curve $f(t), t \in [a, b]$, and the two end-point position vectors $f(a)$ and $f(b)$:



$$P = \frac{1}{2} \int_a^b |x(t)y'(t) - y(t)x'(t)| \, dt.$$

Proof of the area formula

An approximate value of the area is the sum of areas of triangles obtained by subdividing the interval $[a, b]$ into n intervals of length $\Delta t = (b - a)/n$.

The area of a triangle with vertices $(0, 0)$, $f(t_i)$, $f(t_{i+1})$ is

$$\Delta P_i = \frac{1}{2}\|f(t_{i+1}) \times f(t_i)\| \doteq \frac{1}{2}\|(f(t_i) + f'(t_i)\Delta t) \times f(t_i)\|$$
$$= \frac{1}{2}\|f'(t_i) \times f(t_i)\|\Delta t = \frac{1}{2}|y'(t_i)x(t_i) - x'(t_i)y(t_i)|\Delta t,$$

where the last equlatiy follows from the calculation

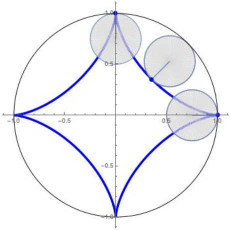
$$\begin{aligned} f'(t_i) \times f(t_i) &= (x'(t_i), y'(t_i), 0) \times (x(t_i), y(t_i), 0) \\ &= (x'(t_i)y(t_i) - y'(t_i)x(t_i), 0, 0). \end{aligned}$$

The area is obtained by adding these and letting $n \to \infty$:

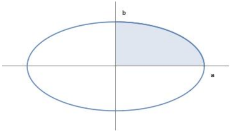
$$\begin{aligned} P &= \lim_{n\to\infty} \frac{1}{2} \sum_{i=0}^{n-1} |y'(t_i)x(t_i) - x'(t_i)y(t_i)|\Delta t \\ &= \frac{1}{2} \int_a^b |x(t)y'(t) - y(t)x'(t)|\, dt. \end{aligned}$$

Problem: the area bounded by

1. the asteroid $x(t) = \cos^3 t, y(t) = \sin^3 t, t \in [0, 2\pi]$ is



2. the elipse $x = a\cos t,\ y = b\sin t, t \in [0, 2\pi]$ is



Hint. In both problems use the identities

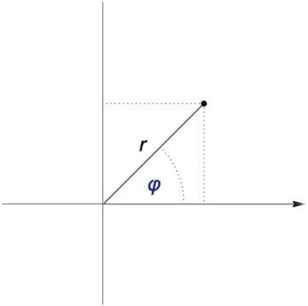
$$\sin^2 t = \frac{1}{2}(1 - \cos(2t)), \qquad \cos^2 t = \frac{1}{2}(1 + \cos(2t)).$$

In the first problem all you have to really integrate after subtractions of some terms is $1 - \cos^2(2t)$. The results are $\frac{3\pi}{8}$ for the first and $ab\pi$ for the second problem.

Curves in the polar plane

Polar coordinates of a point in the plane are

- distance to the origin r, $r \geq 0$, and
- polar angle φ, determined up to a multiple of 2π, defined for $r \neq 0$.



Usually the polar axis corresponds to the positive part of the x-axis, so

- $x = r\cos\varphi$, $y = r\sin\varphi$
- $r = \sqrt{x^2 + y^2}$, $\tan\varphi = \frac{y}{x}$

A curve in polar coordinates is given by $r = r(\varphi), \quad \varphi \in I \subset \mathbb{R}$.

Rule. If $r(\varphi) < 0$, then the point on the curve at an angle φ is equal to

$$(x(\varphi), y(\varphi)) := |r(\varphi)|(\cos\varphi, \sin\varphi) \cdot e^{i\pi}.$$

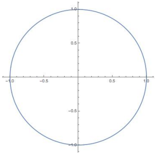
In other words, we reflect the point

$$|r(\varphi)|(\cos\varphi, \sin\varphi)$$

over the origin.

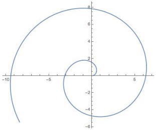
Example

$r = 1$



unit circle

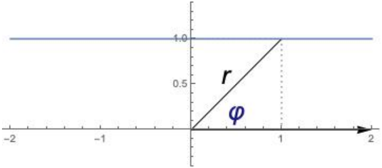
$r = \varphi$



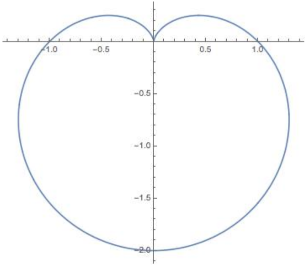
Arhimedean spiral

Example

line $y = 1$, $\quad r = \frac{1}{\sin\varphi}$



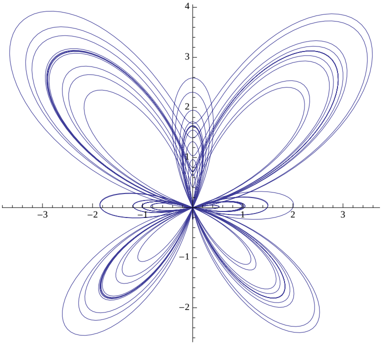
cardioid, $\quad r = 1 - \sin\varphi$



Example

a butterfly

$$r = \sin^5\left(\frac{\varphi - \pi}{12}\right) + e^{\sin\varphi} - 2\cos(4\varphi)$$



Matlab files:

https://zalara.github.io/Algoritmi/curves_polar.m

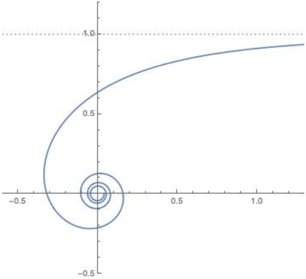
A parametrization of the curve with parameter being the polar angle is:

$$f(\varphi) = \begin{bmatrix} r(\varphi)\cos(\varphi) \\ r(\varphi)\sin(\varphi) \end{bmatrix}, \varphi \in I.$$

Example

The hyperbolic spiral $r = \dfrac{1}{\varphi}$ is parametrized by $f(t) = \begin{bmatrix} \frac{\cos\varphi}{\varphi} \\ \frac{\sin\varphi}{\varphi} \end{bmatrix}$,

$$\begin{aligned} &\text{as } \varphi \to 0, && r(\varphi) \to \infty \\ & && x(\varphi) = \frac{\cos\varphi}{\varphi} \to \infty \\ & && y(\varphi) = \frac{\sin\varphi}{\varphi} \to 1 \\ &\text{as } \varphi \to \infty, && r(\varphi) \to 0 \end{aligned}$$

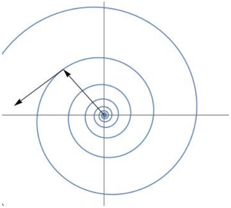


The tangent vector to the curve at a point $r(\varphi)$ is given by

$$f'(\varphi) = \begin{bmatrix} r'(\varphi)\cos(\varphi) - r(\varphi)\sin(\varphi) \\ r'(\varphi)\sin(\varphi) + r(\varphi)\cos(\varphi) \end{bmatrix}$$

Problem: compute the angle between the coordinate vector of a point on the **logarithmic spiral** $r(\varphi) = be^{a\varphi}$ and the tangent vector at that point.

coordinate vector: $f(\varphi) = \begin{bmatrix} be^{a\varphi}\cos(\varphi) \\ be^{a\varphi}\sin(\varphi) \end{bmatrix}$,



tangent vector: $f'(\varphi) = \begin{bmatrix} be^{a\varphi}(a\cos\varphi - \sin\varphi) \\ be^{a\varphi}(a\sin\varphi + \cos\varphi) \end{bmatrix}$,

angle: $\cos\alpha = \dfrac{f(t)\cdot f'(t)}{\|f(t)\|\,\|f'(t)\|} = \dfrac{a}{\sqrt{1+a^2}}$,

so the angle is independent of φ so it is the same at every point on the curve.

Area in polar coordinates

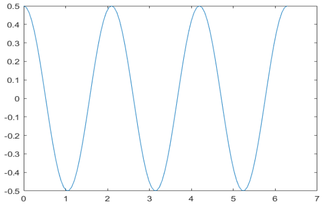
$$P = \frac{1}{2}\int_{\alpha}^{\beta} |xy' - x'y| \, d\varphi = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\varphi$$

Indeed:

$$\begin{aligned} xy' - x'y &= r\cos\varphi(r'\sin\varphi + r\cos\varphi) - r\sin\varphi(r'\cos\varphi - r\sin\varphi) \\ &= r^2(\cos^2\varphi + \sin^2\varphi) = r \end{aligned}$$

Problem: what is the area of one petal of the clover $r(\varphi) = \dfrac{\cos(3\varphi)}{2}$?

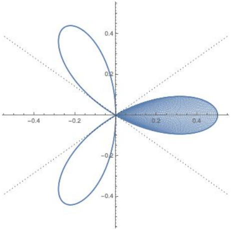
To plot the clover it is convenient to sketch the function $r(\varphi)$ first.



Useful angles are

φ	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	$\frac{6\pi}{6}$	$\frac{7\pi}{6}$	$\frac{8\pi}{6}$	$\frac{9\pi}{6}$	$\frac{10\pi}{6}$	$\frac{11\pi}{6}$	$\frac{12\pi}{6}$
$r(\varphi)$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$

$$P = 2\int_0^{\pi/6} \frac{\cos^2(3\varphi)}{4}\, d\varphi = \frac{\pi}{12}$$



Motion in $\mathbb{R}^3$

Let $\mathbf{r}(t) = f(t)$ be the position vector of a particle in space at time t, $1 \leq t \leq 2$.

Then $\mathbf{v}(t) = \mathbf{r}'(t)$ is its velocity and $\mathbf{a}(t) = \mathbf{r}''(t)$ is its acceleration at time t.

Problem: Let $\mathbf{r}(t) = \begin{bmatrix} t^2 \\ 2t \\ \log t \end{bmatrix}$.

1. Compute its position, velocity and acceleration at time $t = 1$, and the length of its path between $t = 1$ and $t = 2$.
2. If at time $t = 2$ the particle leaves its path and goes off in the tangential direction with constant velocity, where will it be at time $t = 3$? What is the length of its path from $t = 1$ to $t = 3$?

1. Since $\mathbf{r}'(t) = \begin{bmatrix} 2t \\ 2 \\ 1/t \end{bmatrix}$ and $\mathbf{r}''(t) = \begin{bmatrix} 2 \\ 0 \\ -1/t^2 \end{bmatrix}$, the position, velocity and acceleration at $t = 1$ are

$$\mathbf{r}(1) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}(1) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{a}(1) = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

and the length of path

$$\int_1^2 \|\mathbf{r}'(t)\|\, dt = \int_1^2 \sqrt{4t^2 + 4 + (1/t)^2}\, dt = \int_1^2 (2t + 1/t)\, dt =$$

$$[2t^2/2 + \log t]_1^2 = 3 + \log 2$$

2. The tangent line at $t = 2$, and the position at $t = 3$ are:

$$L_2(t) = \begin{bmatrix} 4 \\ 4 \\ \log 2 \end{bmatrix} + (t-2) \begin{bmatrix} 4 \\ 2 \\ 1/2 \end{bmatrix}, L_2(3) = \begin{bmatrix} 8 \\ 6 \\ \log 2 + 1/2 \end{bmatrix}$$

and length of the path along the tangent from $t = 2$ to $t = 3$ is

$$\int_2^3 \|\mathbf{v}(2)\|\, dt = 9/2,$$

so the total length is $\log 2 + 7 + \frac{1}{2}$.

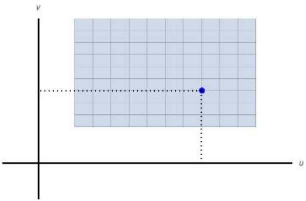
3.4. Parametric surfaces

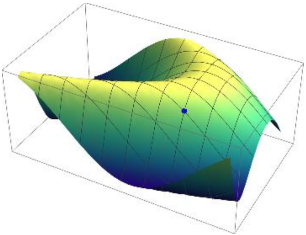
A parametric surface in $\mathbb{R}^m$ is given by a continuous vector function

$$f : D \to \mathbb{R}^m, \qquad D \subset \mathbb{R}^2.$$

We will consider the case $m = 3$:

$$\begin{bmatrix} u \\ v \end{bmatrix} \in D \qquad\qquad f(u, v) = \begin{bmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{bmatrix} \in \mathbb{R}^3$$

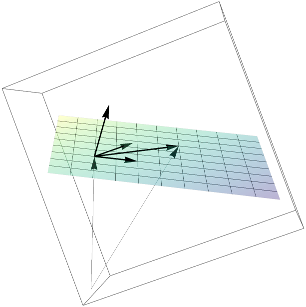




Example

1. A parametric plane through a given point $\mathbf{r}_0 \in \mathbb{R}^3$ with given (noncolinear) vectors $\mathbf{e}_1$ and $\mathbf{e}_2$:

$$f(u, v) = \mathbf{r}_0 + u\mathbf{e}_1 + v\mathbf{e}_2, \quad u, v \in \mathbb{R},$$



The normal to the plane is $\mathbf{n} = \mathbf{e}_1 \times \mathbf{e}_2 \neq 0$.

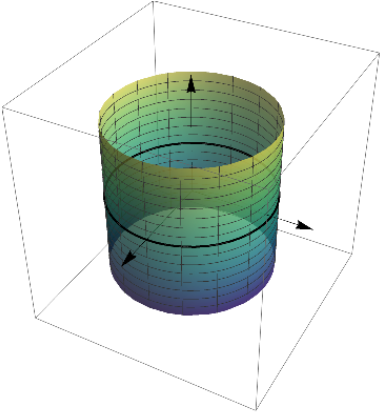
The equation the plane: $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$

Matlab file:

https://zalara.github.io/Algoritmi/plane.m

2.

$$f(u,v) = \begin{bmatrix} \cos u \\ \sin u \\ v \end{bmatrix}, \quad u \in [0, 2\pi], v \in [0,1]$$



a cylinder with radius 1 and axis the z-axis

Matlab file:

https://zalara.github.io/Algoritmi/cylinder.m

For every point $f(u_0, v_0)$ on the surface there are two coordinate curves through it:

- $f(u_0, v)$,
- $f(u, v_0)$,

both lie on the surface.

Example

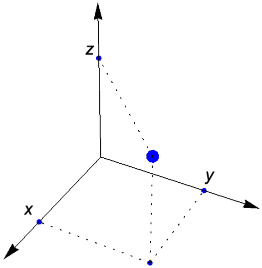
1. In the parametrized plane $f(u, v) = r_0 + ue_1 + ve_2$, $e_1 \times e_2 \neq 0$, coordinate curves are lines parallel to e_2 for a fixed $u = u_0$ and to e_1 for a fixed $v = v_0$.
2. In the cylinder, coordinate curves $u = u_0$ are vertical lines, and $v = v_0$ are circles.

Coordinate systems in $\mathbb{R}^3$

The parameters u and v in surface parametrizations often have a geometric meaning.

For example, they could be two coordinates from one of the standard coordinate systems in $\mathbb{R}^3$:

Cartesian coordinates x, y, z (we know these well)



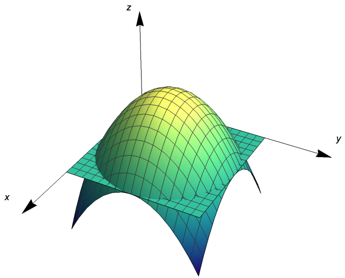
Example

$$f(x,y) = \begin{bmatrix} x \\ y \\ 1-(x-1)^2-(y-1)^2 \end{bmatrix},\ 0 \leq x, y \leq 2$$

The surface is the graph
$z = 1-(x-1)^2-(y-1)^2$,

Coordinate curves:
intersection with planes

$x = x_0$ and $y = y_0$



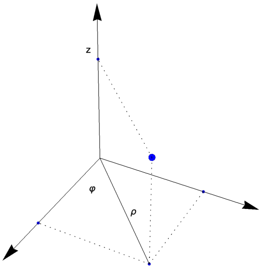
Matlab file:

https://zalara.github.io/Algoritmi/surfaces_coordinate_curves.m

Cylindrical coordinates:

$\rho \geq 0$ distance from z axis, polar radius in plane $z = 0$

φ polar angle in plane $z = 0$



Conversion to cartesian coordinates: $x = \rho \cos \varphi, y = \rho \sin \varphi, z = z$

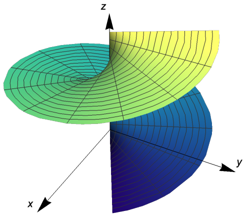
Example

$$f(u,v) = \begin{bmatrix} u\cos v \\ u\sin v \\ v \end{bmatrix}$$

Coordinate curves:

$u = u_0$: helix with radius u_0

$v = v_0$: ray from z-axis with polar angle and height v_0



Matlab file:

https://zalara.github.io/Algoritmi/cylindrical_coordinates_helix.m

Spherical coordinates: r, φ, ψ, where

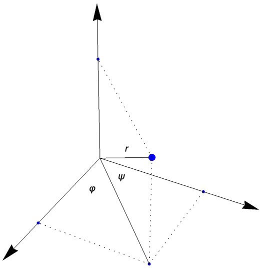
r, $r \geq 0$: distance to the origin,

φ: polar angle in plane $z = 0$

ψ, $-\pi/2 \leq \psi \leq \pi/2$: azimuthal angle between the coordinate vector and plane $z = 0$,

$\psi = \pi/2$: positive part of z axis
$\psi = 0$: plane $z = 0$

$\psi = -\pi/2$ negative part of z-axis



Conversion to cartesian coordinates: $x = r\cos\varphi\cos\psi$, $y = r\sin\varphi\cos\psi$, $z = r\sin\psi$

Conversion to cylindrical coordinates: $\rho = r\cos\psi$, $z = r\sin\psi$

Example

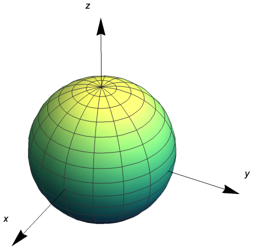
$$f(u,v) = \begin{bmatrix} \cos u \cos v \\ \sin u \cos v \\ sinv \end{bmatrix}, 0 \leq u \leq 2\pi, -\pi/2 \leq v \leq \pi/2$$

The surface is the unit sphere $r = 1$

Coordinate curves:

$u = u_0$: latitude $u = u_0$

$v = v_0$: longitude $v = v_0$



Matlab file:

https://zalara.github.io/Algoritmi/spherical_coordinates.m