Mathematical modelling

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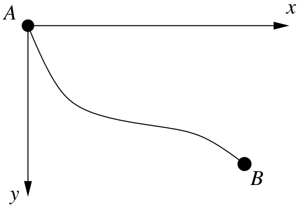
Brachistochrone problem

Problem: Given points A and B what is the fastest path of a mass starting in A and ending in B, being accelerated only by gravity? We assume no friction is present.



- We denote $A = (0,0)$ and $B = (b,0)$. We are searching for a curve
$$y(x) : [0, b] \to \mathbb{R}.$$
- Law of conservation of energy:
$$\text{Potential energy} + \text{Kinetic energy}=\text{constant}$$
$$\frac{1}{2}mv(x)^2 = mgy(x) \quad \Rightarrow \quad v(x) = \sqrt{2gy(x)}.$$

- Let $s(x)$ be the arc length of the curve from A to $(x, y(x))$. We have:

$$s(x) = \int_0^x \sqrt{1 + y'(x)^2} dx$$

and hence

$$s'(x) = \frac{ds}{dx} = \sqrt{1 + y'(x)^2}.$$

- Let $T(y)$ be the travel time along the curve $\{(x, y(x)) \colon x \in [0, b]\}$. We have:

$$T(y) = \int_0^{T(y)} dt = \int_0^{s(b)} \frac{ds}{v(s)} = \int_0^b \frac{\sqrt{1 + y'(x)}}{\sqrt{2gy(x)}} dx.$$

- We need to minimize the functional $T(y) : C[0, b] \to \mathbb{R}$ on the vector space of continuous functions on $[0, b]$.

Theorem (Euler-Lagrange equation)

If y^* is the solution of the minimization problem $\min_{y \in C[0,b]} T(y)$*, then it satisfies the equation*

$$\frac{\partial}{\partial y} f(x, y(x), y'(x)) = \frac{d}{dx} \frac{\partial}{\partial y'} f(x, y(x), y'(x)).$$

Applying Euler-Lagrange equations for the brachistochrone problem, we come to the differential equation

$$y' = \sqrt{\frac{C-y}{y}} \quad \text{for some constant } C.$$

Separation of variables:

$$\sqrt{\frac{y}{C-y}}dy = dx.$$

Integrating both sides and using the substitution $y = C\sin^2(t)$ we get

$$x(t) = C\left(t - \frac{1}{2}\sin 2t\right),\ y(t) = C\left(\frac{1}{2} - \frac{1}{2}\cos 2t\right),$$

which is the cycloide.

For those who want to know more:

https://wiki.math.ntnu.no/_media/tma4180/2015v/calcvar.pdf

https://www.youtube.com/watch?v=Cld0p3a43fU

Curvature

1. Intuitively we would like to measure for what amount does the curve deviate from being the straight line.
2. For the circle of radius R we would like that the curvature is proportional to $1/R$.

The curvature $\kappa(t)$ of a smooth curve $f(t)$ at a point $t = a$ is the rate of change of the unit tangent vector $T(t) = \dfrac{f'(t)}{\|f'(t)\|}$:

$$\kappa(t) = \left\| \frac{1}{ds/dt} T'(t) \right\|.$$

If the curve is parametrized by the arc length s, i.e., $\|f'(s)\| = 1$, then this is simply

$$\kappa(s) = \left\| f''(s) \right\|$$

Problem: what is the curvature of a circle with radius a?

The natural parametrization of the circle is $f(s) = \begin{bmatrix} a\cos(s/a) \\ a\sin(s/a) \end{bmatrix}$, so

$$f'(s) = \begin{bmatrix} -\sin(s/a) \\ \cos(s/a) \end{bmatrix} \qquad \text{and} \qquad f''(s) = \begin{bmatrix} -\cos(s/a)/a \\ -\sin(s/a)/a \end{bmatrix}.$$

The curvature

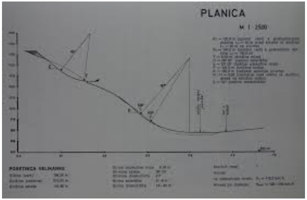
$$\kappa(s) = \|f''(s)\| = 1/a$$

is constant along the circle.

- As $a \to \infty$, the circle goes towards a line and $\kappa \to 0$.
- On the other hand, as $a \to 0$, the circle goes towards a point and $\kappa \to \infty$.

Problem: designing roads and railways

Roads, railway bends, roller coaster loops, the ski jump in Planica . . . are designed so that the transitions from the straight to the circular parts are as smooth as possible.

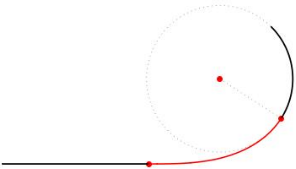


The force acting on a moving point on the curve (car, train, ski jumper,. . .) increases and decreases as evenly as possible.

The transition curve from

- the straight part (with curvature 0) to
- the circular part (with curvature $a > 0$)

has several names: clotoid, Euler spiral, Cornu spiral . . .



Its characteristic property is that the curvature $\kappa(s)$ is a linear function of arc length s.

Let us find its arc length parametrization $f(s)$. Assume that $\kappa(s) = \|f''(s)\| = 2s$.

Remember that the arc length parametrization is the unit speed parametrization, so $\|f'(s)\| = 1$ and so $f'(s)$ can be written in the form

$$f'(s) = \begin{bmatrix} x'(s) \\ y'(s) \end{bmatrix} = \begin{bmatrix} \cos\varphi(s) \\ \sin\varphi(s) \end{bmatrix}.$$

This gives

$$\kappa(s) = \sqrt{x''(s)^2 + y''(s)^2} = \varphi'(s) = 2s, \quad \varphi(s) = s^2,$$

$$x'(s) = \cos(s^2), \quad y'(s) = \sin(s^2),$$

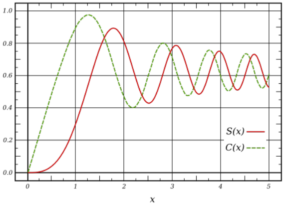
so

$$x(s) = \int_0^s \cos(u^2)\,du, \quad y(s) = \int_0^s \sin(u^2)\,du$$

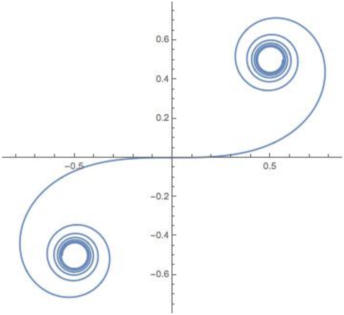
The functions

$$x(s) = \int_0^s \cos(u^2)\,du = C(s), \quad y(s) = \int_0^t \sin(u^2)\,du = S(s)$$

are nonelementary functions called the Fresnel integrals



Fresnel integrals



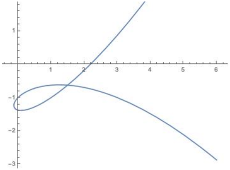
clotoid

Plane curves

For a plane curve $f(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ the tangent at a regular point $f(a)$ is

- the vertical line
$$x = x(a)$$
if $x'(a) = 0$ and $y'(a) \neq 0$,
- the line
$$y - y(a) = \frac{y'(a)}{x'(a)}(x - x(a))$$
if $x'(a) \neq 0$,
- the horizontal line
$$y = y(a)$$
if $y'(a) = 0$ and $x'(a) \neq 0$.

Plotting a parametric plane curve

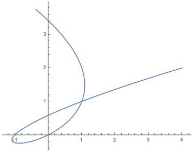


Here is a general strategy:

- find the asymptotic behaviour: $\lim_{t\to\infty} f(t)$, $\lim_{t\to-\infty} f(t)$
- find intersections with coordinate axes: solve $y(t) = 0$ and $x(t) = 0$
- find points where the tangent is vertical or horizontal: solve $x'(t) = 0$ and $y'(t) = 0$
- find self-intersections: solve $f(t) = f(s)$, $t \neq s$
  - and the two tangents there
- look for other helpful features ...
- connect points $\mathbf{r}(t) = f(t)$ by increasing t

Problem: find the self-intersection (if there is one) of a parametric curve

Let $f(t) = \begin{bmatrix} t^3 - 2t \\ t^2 - t \end{bmatrix}$



A self-intersection is at a point $f(t) = f(s)$, with $t \neq s$, so:

$$t^3 - 2t = s^3 - 2s \quad \text{and} \quad t^2 - t = s^2 - s$$
$$\Rightarrow \quad t^3 - s^3 = 2t - 2s \quad \text{and} \quad t^2 - s^2 = t - s$$

Since $t \neq s$ we can divide by $t - s$:

$$t^2 + ts + s^2 = 2 \quad \text{and} \quad t + s = 1$$
$$\Rightarrow \quad t = 1 - s \quad \text{and} \quad (1-s)^2 + s(1-s) + s^2 = 2.$$

The self-intersection (where s and t can be interchanged) is at

$$s = (1+\sqrt{5})/2, \; t = (1-\sqrt{5})/2, \quad f(t) = f(s) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

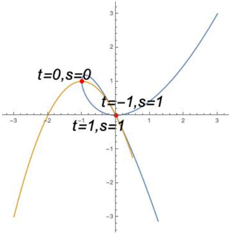
Problem: do two parametric curves intersect. Imagine two cars speeding along the two curves. Do they crash?

Let $f_1(t) = \begin{bmatrix} t^2 - 1 \\ -t^3 - t^2 + t + 1 \end{bmatrix}$, $\quad f_2(s) = \begin{bmatrix} s - 1 \\ 1 - s^2 \end{bmatrix}$.
To find the intersections, solve the system

$$\begin{aligned} t^2 - 1 = s - 1 \quad &\text{and} \quad -t^3 - t^2 + t + 1 = 1 - s^2 \\ \Rightarrow \quad s = t^2 \quad &\text{and} \quad -s^6 - s^4 + s^2 + 1 = 1 - s^2 \end{aligned}$$

There are three solutions:

$$\begin{aligned} t = -1, s = 1 \quad &\Rightarrow \quad x = 0, y = 0 \\ t = 0,\ s = 0 \quad &\Rightarrow \quad x = -1,\ y = 1 \\ t = 1,\ s = 1 \quad &\Rightarrow \quad x = 0,\ y = 0 \end{aligned}$$



The cars meet at $t = 0, s = 0$ at the point $(-1, 1)$ and at $t = 1, s = 1$ at the point $(0, 0)$.

Problem: plot $f(t) = \begin{bmatrix} t^2 - 1 \\ -t^3 - t^2 + t + 1 \end{bmatrix}$, $f'(t) = \begin{bmatrix} 2t \\ -3t^2 - 2t + 1 \end{bmatrix}$

- Asymptotic behaviour: $\lim\limits_{t\to\infty} f(t) = \begin{bmatrix} \infty \\ -\infty \end{bmatrix}$, $\lim\limits_{t\to-\infty} f(t) = \begin{bmatrix} \infty \\ \infty \end{bmatrix}$,
  - intersections with axes: $t = \pm 1$, at $(0,0)$
    this is also a self-intersection
  - the two tangent lines at $(0,0)$
    - at $t = -1$: $y = 0$,
    - at $t = 1$: $y = -2x$
  - vertical tangent: $t = 0$ at $(-1,1)$
  - horizontal tangent
    - at $t_1 = -1$, $y = 0$,
    - at $t_2 = 1/3$, $y = 32/27$