Mathematical modelling

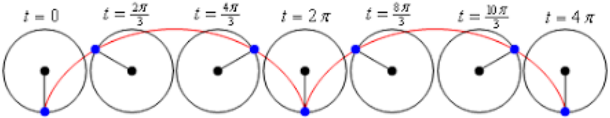
Lecture 6, March 22, 2022

Faculty of Computer and Information Science
University of Ljubljana

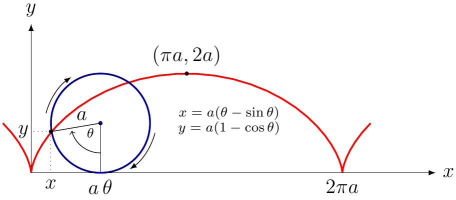
2021/22

Problem: What path does the valve on your bicycle wheel trace as you bike along a straight road?

Represent the wheel as a circle of radius *a* rolling along the *x*-axis, the valve as a fixed point on the circle, the parameter is the angle of rotation:

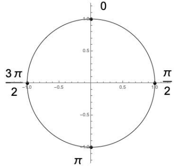


The curve is a cycloid: $x(\theta) = a\theta - a\sin\theta, y(\theta) = a - a\cos\theta$.

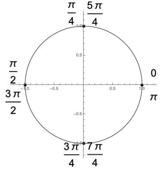


The following parametric curves all describe the circle with radius a around the origin (as well as many others):

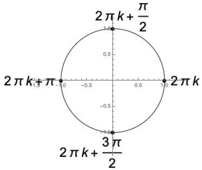
$$f_1(t) = \begin{bmatrix} a \sin t \\ a \cos t \end{bmatrix},\ t \in [0, 2\pi]$$



$$f_2(t) = \begin{bmatrix} a \cos 2t \\ a \sin 2t \end{bmatrix},\ t \in [0, 2\pi]$$



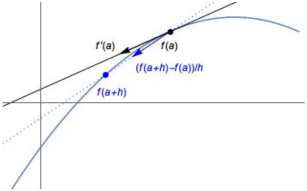
$$f_3(t) = \begin{bmatrix} a \cos t \\ a \sin t \end{bmatrix},\ t \in \mathbb{R}$$



Derivative, linear approximation, tangent

The derivative of the vector function $f(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{bmatrix}$ at the point a is the vector:

$$Df(a) = \begin{bmatrix} x_1'(a) \\ \vdots \\ x_m'(a) \end{bmatrix} = f'(a) = \lim_{h \to 0} \frac{1}{h}(f(a+h) - f(a))$$

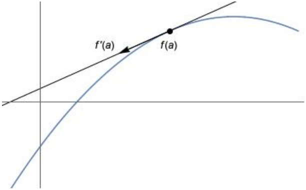


The vector $f'(a)$ (if it exists) represents the velocity vector of a point moving along the curve at the point $t = a$.

If $f'(a) \neq 0$ it points in the direction of the tangent at $t = a$.

The linear approximation of the function f at $t = a$ is

$$L_a(t) = f(a) + (t - a)f'(a)$$



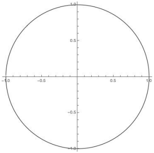
- If $f'(a) \neq \mathbf{0}$, this is a parametric line corresponding to the tangent line to the curve $f(t)$ at $t = a$. In this case $f(a)$ is a regular point of the parametrization.

- If $f'(a) = \mathbf{0}$ (or if it does not exist), the parametrization of the curve is singular in the point $f(a)$.

- A curve $C \in \mathbb{R}^m$ is smooth at a point P on C if there exists a parametrization $f(t)$ of C, such that $f(a) = P$ and $f'(a) \neq 0$.

- A smooth curve has a tangent at every point $P \in C$.

Problem: Is the curve $C = \{f(t), t \in [0, \sqrt{2\pi}]\}$, $f(t) = \begin{bmatrix} \cos(t^2) \\ \sin(t^2) \end{bmatrix}$, smooth?



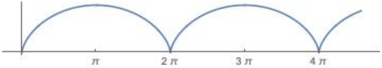
Since $x^2 + y^2 = 1$, $f(t)$ is a parametrization of the unit circle which is a smooth curve (it has a tangent at every point).

Since $f'(0) = \mathbf{0}$ the parametrization f is singular in the point $(1, 0)$.

However, a smooth parametrization exists. Can you find it?

Problem: Is the cycloid a smooth curve?

Our parametrization



$$f(t) = \begin{bmatrix} t - \sin t \\ 1 - \cos t \end{bmatrix}, \quad f'(t) = \begin{bmatrix} 1 - \cos t \\ \sin t \end{bmatrix}$$

is not smooth at $t = 2k\pi$ since $f'(2k\pi) = \mathbf{0}$.

Does a tangent exist?
The slope of the tangent line at a point $f(t)$ is:

$$k_t = \frac{y'(t)}{x'(t)} = \frac{\sin t}{1 - \cos t}$$

The left and right limits as $t \to 2k\pi$ are

$$\lim_{t \nearrow 2k\pi} k_t = \lim_{t \nearrow 2k\pi} \frac{\cos t}{\sin t} = -\infty, \quad lim_{t \searrow 2k\pi} k_t = \lim_{t \searrow 2k\pi} \frac{\cos t}{\sin t} = \infty,$$

so at these points the curve forms a sharp spike (a cusp) and a tangent does not exist.

So, the cycloid is not smooth at the points where it touches the x axis.

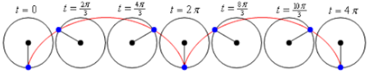
(l'Hospital's rule was used to compute the limits.)

Arc length and the natural parametrization

The arc length s of a parametric curve $f(t)$, $t \in [a, b]$, in $\mathbb{R}^m$ is the length of the curve between the points $t = a$ in $t = b$, i.e. the distance covered by a point moving along the curve between these two points.

Example

For example, what distance does a point on the circle cover when the circle makes one full turn?



Proposition

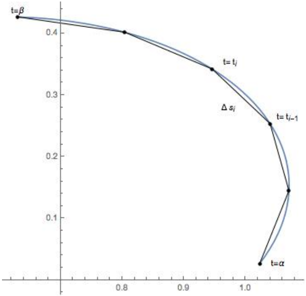
The arc length s of a parametric curve $f(t)$ between the points $t = a$ and $t = b$ is given by

$$s = \int_a^b \|f'(t)\| \, dt.$$

Proof of the Proposition

An aproximate value for s is the length of a polygonal curve connecting close enough points on the curve:

$$
\begin{aligned}
s_n &= \sum_{i=1}^{n} \|f(t_i) - f(t_{i-1})\| \\
&= \sum_{i=1}^{n} \|f'(t_{i-1})\| \Delta t \\
&\to_{n\to\infty} \int_a^b \|f'(t)\|\, dt
\end{aligned}
$$



where:

- The value $f(t_i) = f(t_{i-1} + \Delta t)$, where $\Delta t = t_i - t_{i-1}$, was approximated as $f(t_i) = f(t_{i-1}) + f'(t_{i-1})\Delta t$ and hence $f(t_i) = f(t_{i-1}) + f'(t_{i-1})\Delta t$. (Under the assumption that f' is continuous.
- In the last line we used that the sum represents a Riemannian sum of the function $\|f'(t)\|$.
- For n big enough, s_n is a practical approximation for s.

Problem: The length of the path traced by a point on the circle after a full turn?

A parametrization is $f(t) = \begin{bmatrix} t - \sin t \\ 1 - \cos t \end{bmatrix}$ and hence:

$$\begin{aligned} s &= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\, dt = \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt = \int_0^{2\pi} \sqrt{4\sin^2(t/2)}\, dt \\ &= \int_0^{2\pi} 2\sin(t/2)\, dt = -4(\cos(\pi) - \cos(0)) = 8. \end{aligned}$$

Problem: What is the arc length of the helix $f(t) = \begin{bmatrix} a\cos t \\ a\sin t \\ bt \end{bmatrix}$, $0 \le t \le 2\pi$?

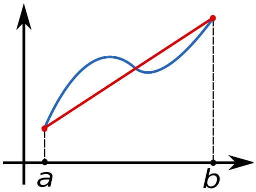
Problem: The circumference of the elipse $\begin{bmatrix} a\cos t \\ b\sin t \end{bmatrix}$, $a \neq b$?

$$\int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\, dt = 4a\int_0^{\pi/2} \sqrt{1 - e^2\sin^2 t}\, dt = 4a\mathrm{E}(e)$$

where $e = \sqrt{1-(b/a)^2}$ is its eccentricity and the function E is the nonelementary elliptic integral of 2nd kind. It can be computed numerically, which is briefly explained in the next few slides.

Numerical integration

The integral $\int_a^b f(x)\,dx$ can be approximated by a linear approximation of f over the interval $[a, b]$ and computing the area of the trapezoid formed.



$$\boxed{\int_a^b f(x)\,dx \approx f(a) + \frac{f(b) - f(a)}{b - a}(x - a) =: T(b - a)}$$

Of course the error of this approximation is usually large and we are not satisfied. How do we estimate how good is this approximation?

Adaptive trapezoid rule (*integral*$(\cdots)$ in Matlab)

1. $T(b-a) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$.

2. We add another point in the middle of the interval, i.e., $x = \frac{a+b}{2}$ and compute the sum of the areas of two trapezoids formed:

$$T((b-a)/2) = \frac{1}{2}T(b-a) + \frac{b-a}{2} \cdot f((a+b)/2).$$

3. If $e := |T(b-a) - T((b-a)/2)|$ is smaller than the tolerance *tol*, we are satisfied and return $T((b-a)/2)$.

4. Otherwise we have to repeat the procedure on each of the subintervals $[a, (a+b)/2]$ and $[(a+b)/2, b]$, where the tolerance on each of them must be smaller than $tol/2$.

5. We can implement this recursively, obtaining the so called adaptive trapezoid rule, where on different subintervals of $[a, b]$ different number of recursions is needed (this depends on the behaviour of the function f).

Natural parametrization

The arc length from the initial $t = a$ to an arbitrary $t = T$

$$s(t) = \int_a^t \|f'(u)\|\, du$$

is an increasing function of t if f is a smooth parametrization, so it has an inverse

$$t(s) : [0, s(T)] \to [a, T].$$

So, the original parameter t can be expressed as a funcion of the arc length s.

Inserting this into the parametrization gives the same curve with a different parametrization:

$$g(s) = f(t(s)).$$

The arc length s is called the natural parameter of the curve.

Proposition

A curve C is parametrized with the natural parameter s satisfies

$$\|g'(s)\| = 1, \tag{1}$$

i.e., the length of the velocity vector is 1 at every point and so a parametrization with the natural parameter is the unit speed parametrization.

Proof. Indeed,

$$g'(s) = \frac{dg}{ds}(s) = \frac{d(f \circ t)}{ds}(s) = \frac{df}{dt}(t(s)) \cdot \frac{dt}{ds}(s) = f'(t(s))t'(s). \tag{2}$$

Now note that by the fundamental theorem of calculus we have that

$$s'(t) = \|f'(t)\|$$

and hence

$$t'(s) = \frac{1}{\|f'(t(s))\|}.$$

Plugging this into (2) we get

$$g'(s) = \frac{f'(t(s))}{\|f'(t(s))\|},$$

which is equivalent to (1).

Example

The standard parametrization of the circle

$$f(t) = \begin{bmatrix} a\cos t \\ a\sin t \end{bmatrix}$$

is not the natural parametrization if $a \neq 1$, since

$$\|f'(t)\| = \sqrt{a^2\cos^2 t + a^2\sin^2 t} = a \neq 1.$$

Since

$$s(t) = \int_0^t a\,dt = at,$$

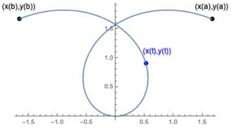
it follows that $t = s/a$ and the natural parametrization is

$$g(s) = \begin{bmatrix} a\cos(s/a) \\ a\sin(s/a) \end{bmatrix}.$$

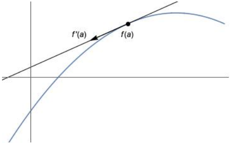
Remember:

A parametric curve: $f(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{bmatrix}$,

$t \in I \subset \mathbb{R}$,



The derivative $f'(t) = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_m'(t) \end{bmatrix}$ is the velocity vector or tangent vector if $f'(t) \neq 0$,



The image $C = \{f(t), t \in I\}$: a (geometric) curve in $\mathbb{R}^m$.A curve C has many parametrizations.

The arc length parametrization or natural parametrization $f(s)$:
s is the length of the chord from $f(a)$ to $f(s)$, $\|f'(s)\| = 1$.