

1. Let

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & -1 & -2 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix}, \quad A' = \begin{bmatrix} -1 & 2 \\ 1 & -2 \\ 1 & 2 \\ -1 & -2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 3 \\ -3 \\ 2 \\ 0 \end{bmatrix}.$$

- (a) Is the system $A\mathbf{x} = \mathbf{b}$ solvable? Is the system $A'\mathbf{x} = \mathbf{b}$ solvable? Find orthogonal projections \mathbf{b}_1 and \mathbf{b}'_1 of the vector \mathbf{b} onto $C(A)$ and $C(A')$, and then find all the solutions of the systems $A\mathbf{x} = \mathbf{b}_1$ and $A'\mathbf{x} = \mathbf{b}'_1$.
- (b) Find the singular value decomposition of A ; $A = USV^T$. This can be obtained using the eigenvalue decomposition of $A^T A$.
- (c) Find the Moore–Penrose pseudoinverses of A and A' , and evaluate $A^+\mathbf{b}$ and $A'^+\mathbf{b}$. Explain the result.
- (d) Solve the exercise in octave, using the commands `svd(A)` and `pinv(A)`.

2. **SVD and image compression.** A greyscale image can be represented by a matrix A . (A color image can be represented using three matrices, say A_R , A_G and A_B). Using the matrices U , S , and V from the SVD decomposition we can reconstruct the matrix A by computing USV^T . Moreover, we can decide that small singular values contribute very little to the image and can be ignored. Let S' be the matrix that contains the largest m singular values on the diagonal. Then $A' = US'V^T$ can serve as an approximation to A .

- (a) Download the image `lena512.mat` and use `A = imread("lena512.mat")` to load it into octave/Matlab. To show the image use `imshow(A)`.
- (b) Find the SVD decomposition of A .
- (c) Compute the approximations for A obtained by using 10, 20, 50, 100 of the largest singular values of A . Show the images and visually assess the quality of the images.
- (d) How much space would we actually need to save such an approximation?