Discrete Morse Theory

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Homology and persistent homology detect holes in spaces through the use of algebraic constructions: a simplicial complex generates a chain complex and the resulting homology construction detects holes. However, functions and vector fields also contain information about the topology of the domain. In the smooth setting this information is contained in critical points of functions and zeros of vector fields, a situation which is beautifully described by the Morse Theory.

In this chapter we will describe discrete Morse Theory. As the name suggests we will delve into the world of discrete functions and discrete vector fields defined on simplicial complexes. Our main goal will be to describe how these encode homology, often leading to simplified representations and faster computations than the standard methods.

1 Motivation

We first recall the definition of elementary collapses.

Definition 1.1. *A simplex in a simplicial complex is a* ***free face*** *if it is a face of precisely one simplex. This implies that the coface in question is a maximal simplex*

Let K be a simplicial complex, $\sigma^{(k-1)} \subset \tau^{(k)} \in K$, and assume σ is a free face in K. A removal $K \to K \setminus \{\sigma, \tau\}$ is called an ***elementary collapse****.*

Complex K is ***collapsible to a subcomplex*** *$L \leq K$ if there is a* ***collapse*** *(i.e., a sequence of elementary collapses) resulting in the subcomplex L. Complex K is* ***collapsible*** *if it is collapsible to a point.*

☞ If σ is a free face in a simplicial complex K then its only coface is a maximal simplex in K.

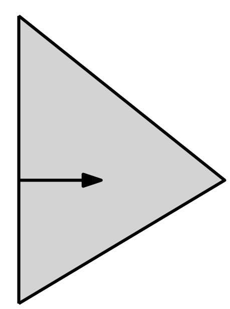

Figure 1: An elementary collapse indicated by an arrow from σ into τ.

Remark 1.2. *We have already proved that an elementary collapse results in a homotopically equivalent space. As a result, if a simplicial complex K is collapsible to a subcomplex $L \leq K$ then $L \simeq K$. In particular, each collapsible simplicial complex is contractible. The converse does not hold as there exist contractible simplicial complexes without a free face, for example Dunce hat (Figure 2) and Bing's house.*

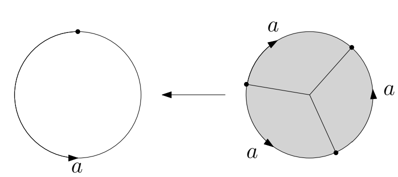

Figure 2: Dunce hat is obtained by glueing the boundary of a disc along a circle: twice alone one direction and once along the other direction. The obtained space can be triangulated but contains no free face meaning it is not collapsible. However, it turns out to be contractible.

Given a simplicial complex K it would be of interest to simplify (i.e. collapse) it as much as possible. This would, for example, simplify the computation of homology groups. One would go about such simplification by repeating the following sequence as long as possible: find a free face and perform the corresponding collapse. An example is given in Figure 3, where the collapses of the first three steps are indicated by the arrows. One can encode such a collapse by:

- drawing all the arrows[1] indicating collapses, or
- annotate simplices by numbers[2] so that the countdown-sequence encodes[3] the collapsing sequence.

Both of these encodings are demonstrated on the right side of Figure 3.

[1] An example of a discrete vector field.

[2] A rough example of a discrete Morse function.

[3] 10 collapses to 9; 8 collapses to 7, etc.

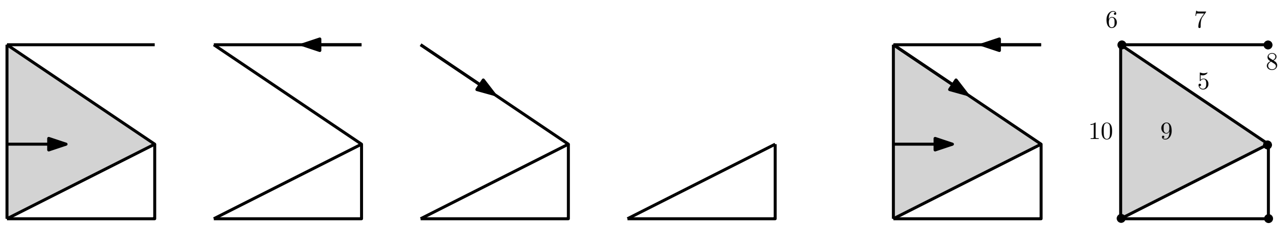

Figure 3: A simplification of a simplicial complex using elementary collapses and the encoding of the resulting collapse by arrows (discrete vector field) and annotations (discrete Morse function).

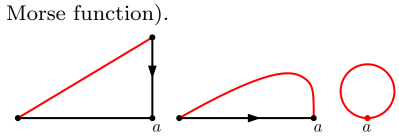

Figure 4: Declaring the red edge to be critical, we can collapse the other two edges and obtain a representation of a circle using only two critical "simplices".

Eventually a collapsing sequence ends when there are no more free faces. At this point we can resort to another trick that will on one hand change the structure of a complex, yet still simplify its description in a way. Choose any simplex, declare it to be a critical simplex, remove it from the complex, and continue with collapsing. In the end we will form a "complex" consisting of critical simplices. The details of the construction will be described throughout this lecture. At this point we only illustrate a geometric interpretation of this idea in terms of "stretching" simplices.

For our motivational purposes let us continue in Figure 4 with the example from Figure 3. We are left with a triangle. We choose one of its edges to be a critical edge and continue with collapsing. We can imagine that each collapse stretches the critical edge until, at the end, we are left with two critical simplices: and edge and a point jointly forming a circle. The resulting space is homotopy equivalent to our original simplicial complex of Figure 3, has a simple representation, but is not a simplicial complex. However, its homology can be computed in the same way as the simplicial homology so in effect, we have transformed the 1-dimensional boundary matrix from 6 columns to 1 column, etc. For another example see Figure 5.

2 Discrete Morse functions and discrete vector fields

We start by defining functions that encode the collapsing sequences and deformations.

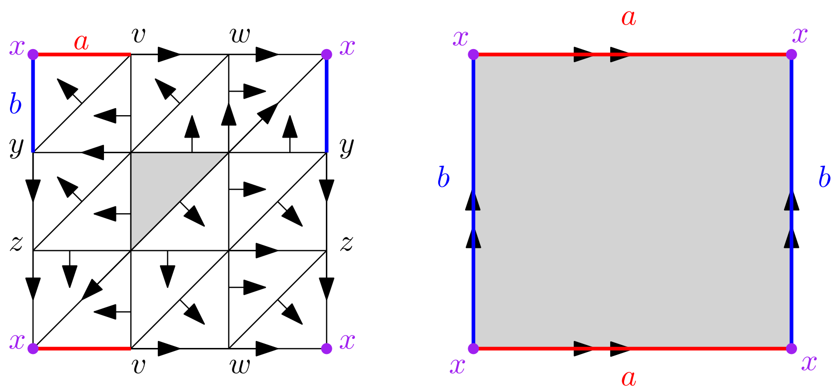

Figure 5: Stretching critical simplices of a standard triangulation of the torus (on the left) along the indicated collapses results in a standard representation of a torus as a square with identified sides (on the right).

Critical simplices can be thought of as zeros of the resulting discrete vector field. We have already seen in the Hairy ball Theorem that there is a connection between zeros of smooth vector fields and topology of the domain.

Definition 2.1. *Let K be an abstract simplicial complex. A function $f\colon K \to \mathbb{R}$ is a **discrete Morse function** [DMF] if $\forall \sigma^{(k)} \in K$:*

1. $e_1 = |\{\tau^{(k-1)} \in K \mid f(\tau) \geq f(\sigma)\}| \leq 1$ *and*
2. $e_2 = |\{\tau^{(k+1)} \in K \mid f(\tau) \leq f(\sigma)\}| \leq 1.$

☞ An abstract simplicial complex is a collection of simplices hence a real function defined on it maps each simplex into a real number.

A function $g\colon K \to \mathbb{R}$ respects dimension[4] if for each $\sigma^{(k-1)} \subset \tau^{(k)} \in K$ we have $g(\sigma) < g(\tau)$. Such a function is a DMF. On the other hand, each DMF almost respects the dimension in the sense[5] that for each simplex $\tau^{(k)}$ at most one exceptional facet and at most one exceptional coface of dimension $k+1$ are allowed. The following proposition demonstrates that the two exceptions cannot occur simultaneously.

[4] As an example think of $g(\sigma) = \mathsf{dim}(\sigma)$.

[5] Putting it differently, for each simplex the values of the function strictly decrease by passing to its faces with at most one exception, and the values of the function strictly increases by passing to its cofaces with at most one exception.

Proposition 2.2. *Given the notation of Definition 2.1 either $e_1 = 0$ or $e_2 = 0$.*

Proof. Aiming for the contradiction assume that for $\sigma \in K$ and for vertices $v_1, v_2 \in K^{(0)}$ we have

$$f(\sigma) \geq f(\sigma \cup \{v_1\}) \geq f(\sigma \cup \{v_1, v_2\}). \tag{1}$$

But then $\sigma \cup \{v_2\} \in K$ and we have:

1. $f(\sigma \cup \{v_2\}) > f(\sigma)$ as the exceptional coface of σ is $\sigma \cup \{v_1\}$.
2. $f(\sigma \cup \{v_2\}) < f(\sigma \cup \{v_1, v_2\})$ as the exceptional face of $\sigma \cup \{v_1, v_2\}$ is $\sigma \cup \{v_1\}$.

These two conclusions combine into $f(\sigma) < f(\sigma \cup \{v_1, v_2\})$ which contradicts equation 1. □

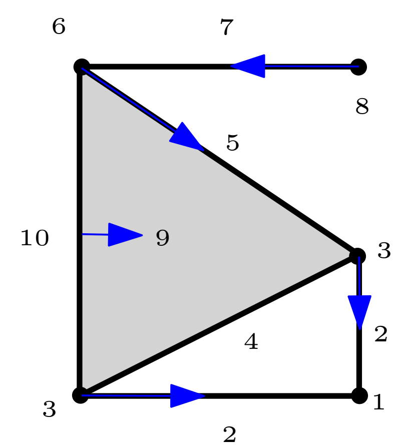

Figure 6: An example of a DMF and the resulting discrete vector field in blue.

Proposition 2.2 implies that simplices with exceptions form disjoint pairs. We will refer to such pairs as **regular pairs**. A regular pair consists of a simplex τ and its facet σ. It encodes an "arrow" $\sigma \to \tau$ in the sense of the motivational section and is thus presented as such, see Figure 6 for an example. A simplex without any exception[6] is called[7] **critical simplex**. Given a DMF on a simplicial complex, each simplex is either critical or contained in a unique regular pair.

[6] I.e., for which $e_1 = e_2 = 0$.

[7] A critical simplex is not contained in any regular pair.

Definition 2.3. *Let K be an abstract simplicial complex. A* ***discrete vector field*** *is a disjoint collection of pairs* (σ_i, τ_i) *of simplices from K such that for each i simplex* σ_i *is a facet of* τ_i. *Each pair of a discrete vector field is referred to as an arrow.*

Proposition 2.4. *Let f be a DMF on a simplicial complex K. For each i let* n_i *denote the number of critical simplices of dimension i. Then* $\chi = n_0 - n_1 + n_2 - \dots$.

Proof. Removing a regular pair of simplices does not change χ because simplices are of adjacent dimensions. □

The disjointness condition means that each simplex can be the member of at most one pair of a discrete vector field. The collection of regular pairs of a DMF forms[8] a discrete vector field, see Figure 6. A discrete vector field is called a **gradient vector field**[9] if it is induced by some DMF in this manner. The arrows constituting a discrete vector field will be sometimes referred[10] to as regular pairs.

[8] The converse is not true in general as we will explain the the next subsection.

[9] We will be omitting adjective "discrete" when mentioning gradient vector fields.

[10] The reason is twofold: to emphasize that the pair is a part of the structure of a discrete vector field, and to stress the analogy with regular pairs of a DMF.

Gradient vector fields

Definition 2.5. *Let K be a simplicial complex and* $p \in \mathbb{N}$. *Given a discrete vector field on K consisting of pairs* $\{(\sigma_i, \tau_i)\}_{i \in J}$, *a* ***p-path*** *is a sequence*

$$\sigma_{i_1}^{(p-1)} \to \tau_{i_1}^{(p)} \geq \sigma_{i_2}^{(p-1)} \to \tau_{i_2}^{(p)} \geq \sigma_{i_3}^{(p-1)} \to \cdots \to \tau_{i_k}^{(p)} \geq \sigma_{i_{k+1}}^{(p-1)}$$

such that for each j:

- $(\sigma_{i_j}^{(p-1)}, \tau_{i_j}^{(p)})$ *is an arrow in the discrete vector field, and*
- $\sigma_{i_j}^{(p-1)}$ *is a facet of* $\tau_{i_{j-1}}^{(p)}$.

  Such a p-path is a ***cycle*** *if* $\sigma_1 = \sigma_{k+1}$ *and* $k \geq 1$.
  A discrete vector field is ***acyclic*** *if it admits no cycle.*

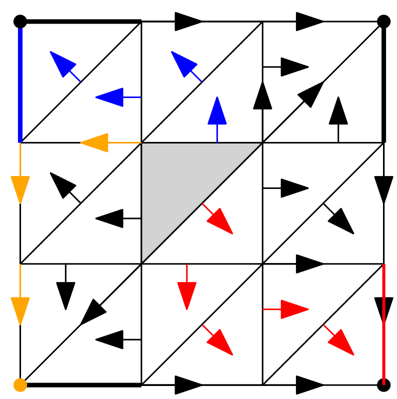

Figure 7: A 2-path in blue ending in a critical edge, a 2-path in red ending in a non-critical edge, and a 1-path in orange ending in a critical vertex.

A few observations concerning Definition 2.5:

1. A critical simplex can only appear as the last simplex of a p-path in a discrete vector field.
2. Given a DMF f, function values decrease along any p-path in the induced discrete vector field, i.e.:

$$f(\sigma_{i_j}) \geq f(\tau_{i_j}) > f(\sigma_{i_{j+1}}), \qquad \forall j.$$

In particular, $f(\sigma_{i_1}) > f(\sigma_{i_m})$ for all $m > 1$.

3. Observation 2. implies that each gradient vector field is acyclic. The following theorem proves the converse.

☞ Different DMFs on a simplicial complex K may induce the same discrete vector field. For example, if f is a DMF, then so are e^f and $3f - 5$, and all of them induce the same discrete vector field. Our primary interest in discrete vector fields lies in their encodings of collapses and deformations-simplifications of a simplicial complex. A DMF represents a convenient but not unique way of encoding a discrete vector field.

Theorem 2.6. *Each acyclic discrete vector field on a simplicial complex K is a gradient vector field, i.e. it is induced by some DMF.*

A proof is given in the appendix. As a result we obtain the following theorem.

Theorem 2.7. *A discrete vector field is gradient vector field iff it is acyclic.*

We conclude this section by demonstrating how acyclic discrete vector fields encode collapses.

Proposition 2.8. *Suppose the critical simplices of an acyclic discrete vector field on K form a subcomplex $L \leq K$. Then there exists a collapse $K \to L$ and thus $K \simeq L$.*

Corollary 2.9. *If an acyclic discrete vector field on K has a single critical simplex, then that simplex is a vertex and K is collapsible.*

Proof. The statement follows directly from Proposition 2.8. □

Proof. We claim there exists a regular pair (σ, τ) such that σ is a free face. Assuming for a moment this claim is true, we can remove pair (σ, τ) by performing an elementary collapse and proceed by using the claim on the resulting complex. Thus the inductive argument and the claim suffice to prove the proposition.

We now turn our attention to proving the claim. Let n denote the maximal dimension of a simplex in $K \setminus L$. There exists an n-path in the discrete vector field. Take a maximal[11] such path and let $\sigma \to \tau$ denote the first regular pair in it. Simplex σ is a free face by the following argument:

- $\sigma \leq \tau$ as the simplices form a regular pair.
- If σ was a facet of another simplex τ' in $K \setminus L$, then n-simplex τ' would be contained in another regular pair[12] (σ', τ'), which could be used to prolong our n-path. This contradicts the maximality of the chosen n-path.
- If σ was a facet of another simplex τ' in L, then $\sigma \in L$ as L is a subcomplex, a contradiction.

□

[11] Such a path contains the first regular pair because the discrete vector field is acyclic.

[12] There are no $(n+1)$-simplices in $K \setminus L$.

3 Morse homology

In this section we will explain the proceedure that leads to the computation of homology from a gradient vector field. The geometric idea behind the theory has been presented at the beginning of this chapter: collapsing regular pairs stretch critical simplices, with the resulting space having the same homotopy type as the original simplicial complex but fewer "simplices". In our treatment we will refrain[13] from formally defining the resulting space and instead construct the resulting chain complex directly. However, it might still be helpful to keep the geometric idea in mind to help navigate the algebraic construction.

[13] A formal definition of resulting spaces would require a significant amount of additional material from algebraic topology. This would include a formal treatment of CW complexes, i.e., spaces obtained by inductively glueing discs. We have actually mentioned several presentations of such constructions when presenting Torus, Klein Bottle and projective plane by drawing a square with identifications along the edges, when defining the dunce hat, and in one of the previous appendices in the context of relative homology.

Morse chain complex

For the rest of this section we fix a simplicial complex K, a gradient vector field on K, and an (algebraic) field $\mathbb{F}$ to provide coefficients algebraic constructions. For each i let n_i denote the number of critical i-simplices.

> **Definition 3.1.** *Let $p \in \{0, 1, \ldots\}$. A **Morse p-chain** is a formal sum $\sum_{i=1}^{n_p} \lambda_i \sigma_i^p$ with $\lambda_i \in \mathbb{F}$ and σ_i^p being an oriented critical simplex of dimension p in K for each i.*
>
> *The p-dimensional **Morse chain group** $\mathfrak{C}_p$ is the vector space consisting of all Morse p-chains.*

☞ As with the usual homology, multiplying an oriented simplex by -1 changes its orientation.

Observe that $\mathfrak{C}_p \cong \mathbb{F}^{n_p}$. In order to obtain a chain complex we also need to define the boundary maps. These are based on oriented[14] paths in discrete vector fields.

[14] Paths in a discrete vector field are directed by definition. The adjective "orientable" refers to the fact that the simplices forming the path are oriented.

> **Definition 3.2.** *Let $p \in \{0, 1, \ldots\}$. An **oriented p-path** from an oriented simplex $\sigma_1^{(p-1)}$ to an oriented simplex $\sigma_{k+1}^{(p-1)}$ is a p-path*
>
> $$\sigma_1^{(p-1)} \to \tau_1^{(p)} \geq \sigma_2^{(p-1)} \to \tau_2^{(p)} \geq \sigma_3^{(p-1)} \to \cdots \to \tau_k^{(p)} \geq \sigma_{k+1}^{(p-1)}$$
>
> *consisting of oriented simplices, such that for each j the orientation induced by τ_j on its facets:*
>
> *1. matches σ_j, and*
>
> *2. does not match σ_{j+1}.*

☞ One could say that the simplices τ_{i_j} in an oriented p-path are oriented consistently along the path.

Given an oriented critical p-simplex τ let $\delta(\tau)$ denote the collection of all of its facets with the induced orientation arising from τ. For each[15] oriented critical $(p-1)$-simplex σ define

$$\alpha_{\tau,\sigma} = \sum_{\sigma' \in \delta(\tau)} |\{ \text{ oriented paths from } \sigma' \text{ to } \sigma \}|$$

[15] Given an oriented critical $(p-1)$-simplex σ observe that $\alpha_{\tau,\sigma}$ counts different paths than $\alpha_{\tau,-\sigma}$.

as the number of oriented p-paths from elements of $\delta(\tau)$ to σ.

Definition 3.3. *The **boundary map** $\mathfrak{d}$ of the Morse chain complex is defined as follows: for each oriented critical p-simplex τ define*

$$\mathfrak{d}_p \tau = \sum_{i=1}^{n_{p-1}} (\alpha_{\tau,\sigma_i} - \alpha_{\tau,-\sigma_i})\sigma_i,$$

where $\sigma_1, \dots, \sigma_{n_{p-1}}$ are critical $(p-1)$-simplices with a fixed orientation.

☞ The oriented paths constituting the boundary map model how arrows stretch the boundary of a p-critical simplex towards critical $(p-1)$-simplices.

Examples will be provided below when demonstrating the computation of Morse homology, see also Figures 8 and 9. It turns out that $\mathfrak{d}^2 = 0$.

Definition 3.4. *The **Morse chain complex** is the chain complex defined as*

$$\cdots \xrightarrow{\mathfrak{d}} \mathfrak{C}_n \xrightarrow{\mathfrak{d}} \mathfrak{C}_{n-1} \xrightarrow{\mathfrak{d}} \cdots \xrightarrow{\partial} \mathfrak{C}_1 \xrightarrow{\mathfrak{d}} \mathfrak{C}_0 \xrightarrow{\mathfrak{d}} 0.$$

Morse homology

We may now define the Morse homology as the homology arising from the Morse chain complex.

Definition 3.5. *Let $p \in \{0, 1, \dots\}$. The **Morse homology** of a gradient vector field on K is defined as*

$$\mathfrak{H}_p(K;\mathbb{F}) = \ker \mathfrak{d}_p / \operatorname{Im} \mathfrak{d}_{p+1}.$$

Theorem 3.6. *The Morse homology is isomorphic to the standard (simplicial) homology:*

$$\mathfrak{H}_p(K;\mathbb{F}) \cong H_p(K;\mathbb{F}).$$

Corollary 3.7 (Weak Morse inequality)**.** *For each p the number of critical p simplices is greater or equal to the corresponding Betti number: $n_p \geq \mathfrak{b}_p$.*

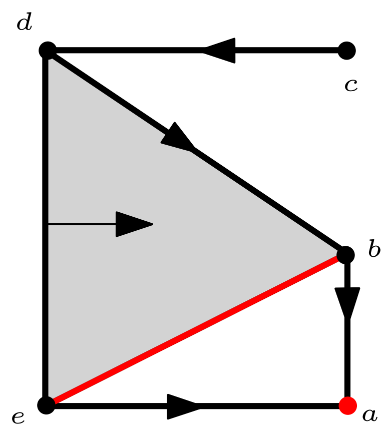

Figure 8: A gradient vector field on a simplicial complex. Critical edges are colored in red. There is one oriented 1-path from $\langle e \rangle$ to $\langle a \rangle$ and one oriented 1-path from $\langle b \rangle$ to $\langle a \rangle$.

Example 3.8. *Given the situation of Figure 8 there is one critical edge $\langle b, e \rangle$ and one critical vertex $\langle a \rangle$. Thus $\mathfrak{C}_1 \cong \mathfrak{C}_0 \cong \mathbb{F}$ with the other Morse chain groups being trivial.*

Let us determine $\mathfrak{d}\langle b, e \rangle$:

1. $\delta(\langle b, e \rangle) = \{\langle e \rangle, -\langle b \rangle\}$.

2. There is one oriented 1-path from $\langle e\rangle$ to $\langle a\rangle$:

$$\langle e\rangle \to \langle e,a\rangle \geq \langle a\rangle.$$

3. In a similar fashion there is one oriented 1-path from $-\langle b\rangle$ to $-\langle a\rangle$.

4. Observations 2. and 3. imply $\alpha_{\langle b,e\rangle,\langle a\rangle} = 1$ and $\alpha_{\langle b,e\rangle,-\langle a\rangle} = 1$.

5. $\partial\langle b,e\rangle = (\alpha_{\langle b,e\rangle,\langle a\rangle} - \alpha_{\langle b,e\rangle,-\langle a\rangle})\cdot\langle a\rangle = 0$

The resulting Morse chain complex is of the form

$$\cdots \to 0 \to \mathbb{F} \xrightarrow{0} \mathbb{F} \to 0.$$

The resulting homology is trivial in dimensions two and above, and nontrivial below: $\mathfrak{H}_0(K) \cong \mathfrak{H}_1(K) \cong \mathbb{F}$.

Example 3.9. *Let us compute the Morse homology of a torus.*

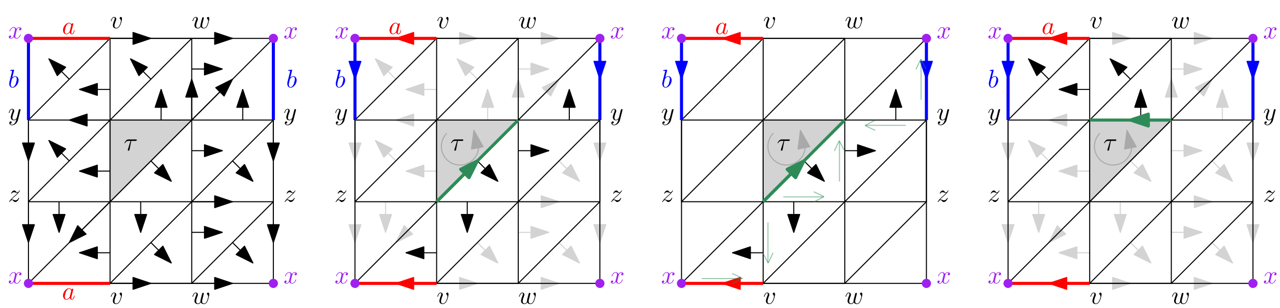

Figure 9: A triangulation of a torus, a gradient vector field, and paths generating the Morse boundary from Example 3.8.

Given the triangulation and the gradient vector field on a torus presented on the leftmost part of Figure 9 we determine the following critical simplices:

- *critical vertex x in purple;*
- *critical edges a (red) and b (blue);*
- *critical triangle τ.*

We orient the critical simplices according to visualizations in the other parts in Figure 9. The resulting Morse chain complex is of the form

$$\cdots \to 0 \to \mathbb{F} \to \mathbb{F}^2 \to \mathbb{F} \to 0.$$

We next determine the Morse boundary of τ. Only two simplices of $\delta(\tau)$ are the starting simplices of oriented 2-chains ending in a critical edge:

- *From the diagonal edge of $\delta(\tau)$ there are two oriented 2-paths[16] to critical edges[17] $-\langle a\rangle$ and $-\langle b\rangle$.*

[16] Drawn in black on the center-left part of Figure 9.

[17] The center-right part of Figure 9 contains opaque green arrows indicating the orientations of the edges contained in the oriented 2-paths. The terminal edges of the oriented 2-paths are $-\langle a\rangle$ and $-\langle b\rangle$.

- *From the top edge of $\delta(\tau)$ there are two oriented 2-paths*[18] *to critical edges*[19] $\langle a \rangle$ *and* $\langle b \rangle$.
- *There are oriented 2-paths starting in the vertical edge of $\delta(\tau)$ but none of them ends in a critical edge.*

Combining these three cases we conclude

$$\mathfrak{d}(\tau) = -\langle a \rangle - \langle b \rangle + \langle a \rangle + \langle b \rangle = 0.$$

In a similar way we conclude that $\mathfrak{d}a = \mathfrak{d}b = 0$. The resulting Morse chain complex is of the form

$$\cdots \to 0 \to \mathbb{F} \xrightarrow{0} \mathbb{F}^2 \xrightarrow{0} \mathbb{F} \to 0.$$

The resulting homology is trivial in dimensions three and above, and nontrivial below: $\mathfrak{H}_0(K) \cong \mathfrak{H}_2(K) \cong \mathbb{F}$ and $\mathfrak{H}_1(K) \cong \mathbb{F}^2$.

[18] Drawn in black on the rightmost part of Figure 9.

[19] The two oriented 2-paths differ only in the last simplex.

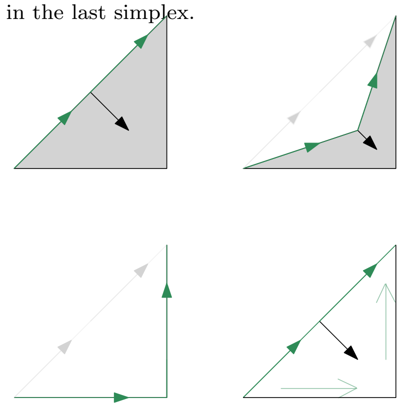

Figure 10: In this figure we provide a geometric justification for the way the orientation carries forward through oriented 2-paths. The bottom-right part is a snapshot from the center-right part Figure 9 indicating how the orientation of the diagonal edge carries on through the arrow to the other two edges of the triangle. The first three parts of this figure indicate how such an orientation on the two edges is obtained by deforming the oriented diagonal edge along the arrow of a discrete vector field.

Generating DMFs and gradient vector fields

Using discrete Morse theory depends on the ability to generate DMFs and gradient vector fields with as few critical simplices as possible. The weak Morse inequalities show that the lower bounds for the numbers of critical simplices are Betti numbers. A DMF on a simplicial complex is **perfect**, if the number of critical p-simplices coincides with the p^{th} Betti number. In terms of the numbers of critical simplices, perfect DMFs are optimal DMFs. Not every simplicial complex admits a perfect DMF: an example is the Dunce hat.

There is a simple algorithm to generate a perfect DMF on a graph. For each component generate a gradient vector field as follows:

- Find a spanning tree.
- Choose a critical vertex.
- Define a gradient vector field pointing towards the critical vertex along the spanning tree, see Figure 11.

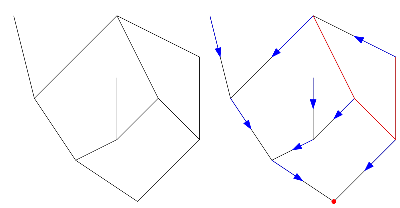

Figure 11: A graph (left) and a blue spanning tree (right) with a gradient vector field pointing to the chosen critical vertex (red). The edges not contained in the tree (red) are the critical edges.

The mentioned construction can be generalized to higher dimensional simplicial complexes: keep adding arrows while making sure that the acyclicity condition is preserved. However, better results are typically obtained through more elaborate designs.

4 Concluding remarks

Recap (highlights) of this chapter

- Discrete Morse functions
- Gradient vector fields
- Morse homology

Background and applications

The smooth Morse theory was developed in the first part of the twentieth century. Amongst its results it relates critical points and gradient flows of a generic function on a manifold to the homology of a manifold. Its discrete version has been introduced at the turn of the millennium. The past two decades saw a considerable development of the discrete Morse theory from various directions, including computational aspects, developing analogies between discrete and smooth results, etc.

An echo of the smooth Morse theory is the Hairy ball theorem: the topology of a domain is connected to zeros of vector fields and thus to extrema of functions. In a similar way, an echo of the discrete Morse theory is our proof of the Euler-Poincaré formula, where we essentially only counted the maxima of the x-coordinate function. Theorem 2.6 is a discrete variant of the assigning of a potential function to a vector field.

Generalized discrete Morse theories can be used to prove[20] that the Cech complexes collapse onto alpha complexes in Euclidean spaces. Several computer programs use discrete Morse theory to a different degree to assist[21] with computations of homology. The theory can also be used as a preprocessing tool or a framework within which to analize discrete functions.

[20] By Bauer and Edelsbrunner.

[21] For example, using simplification using emergent pairs in Ripser. On the other hand Perseus is actually based on a discrete Morse theory.

A proof of Theorem 2.6

We first introduce some preliminary notions. Given a simplicial complex K the **Hasse diagram** of K is a directed graph defined as follows:

1. The nodes are the simplices of K;
2. Directed edges correspond to pairs (simplex, a facet). In particular, each n-simplex is the source of $n+1$ directed edges.

An example is provided in Figure 12. The directed edges in the graph represent the containment of a facet.

Given an empty discrete vector space on a simplicial complex K, the directed edges also represent the direction of descent of the corresponding DMF. In this trivial case all the simplices are critical, there are no exceptions and the DMF respects dimension. An obvious choice of a DMF in this case is the dimension function of a simplex. For illustrative purposes let us discuss how we could obtain such a function from the Hasse diagram in an inductive manner:

- Assign the smallest value, say 0 to all minimal nodes of a directed graph;

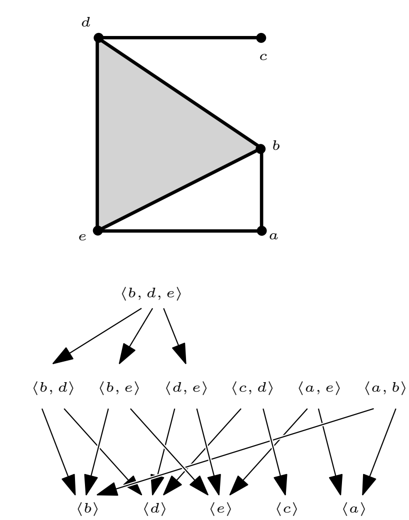

Figure 12: A simplicial complex and its Hasse diagram. Hasse diagrams are typically drawn in levels corresponding to the dimensions of simplices.

- Assign the second smallest value, say 1 to all nodes whose lower set[22] has already been enumerated;
- Proceed by induction: In step number n assign the nth smallest value, say $n-1$ to all nodes whose lower set has already been enumerated;

[22] The lower set of a node σ is the collection of all nodes which appear as the target of a directed edge starting at σ.

This inductive construction of function works for any acyclic directed graph and will be used in our eventual proof.

Given a non-trivial discrete vector field on a simplicial complex K we define a modified Hasse diagram of K by reverting the direction of the directed edges corresponding to the regular pairs, see Figure 12 for an example. A modified Hasse diagram encodes the sufficient conditions on a DMF to generate the initial discrete vector field. The above inductive procedure on such a diagram will produce a required DMF iff the diagram itself is acyclic as a directed graph.

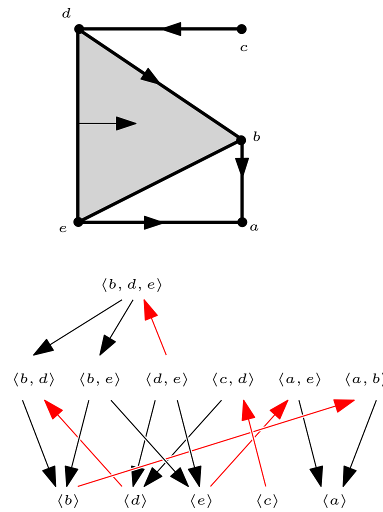

Figure 13: The modified Hasse diagram, the reverted edges are red.

Lemma 4.1. *The modified Hasse diagram of an acyclic discrete vector field is acyclic.*

Proof. Since each simplex can be a member of at most one regular pair in a discrete vector field, a cycle in the modified Hasse diagram H can't contain consecutive directed edges corresponding to regular pairs. As directed edges of H either end in a simplex of dimension 1 higher (in the case of regular pairs) or lower (in the case of edges encoding the facet relation) than the dimension of the initial simplex, any cycle in H has to be an alternating concatenation of these two types. As a result, a cycle in H corresponds to a p-cycle in the initial discrete vector field, which is non-existent by the main assumption. □

A proof of Theorem 2.6. Given an acyclic discrete vector field, the corresponding modified Hasse diagram is acyclic by Lemma 4.1. Thus the inductive procedure above results in a suitable DMF, see Figure 14. □

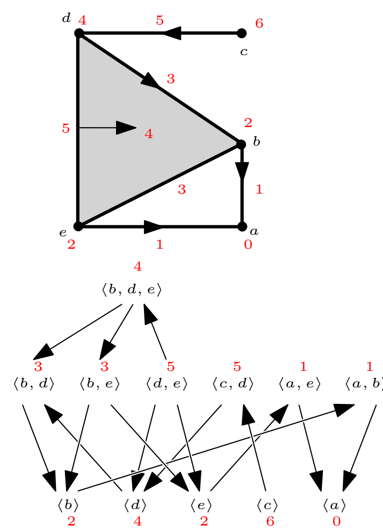

Figure 14: The modified Hasse diagram and the DMF (in red) constructed by the inductive proceedure.