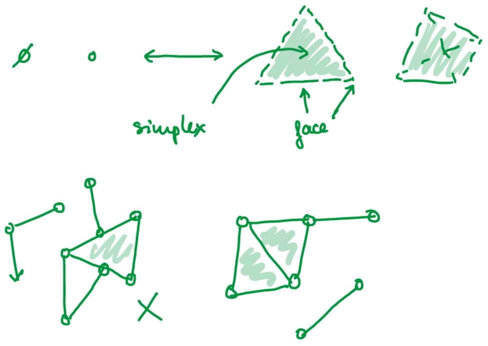
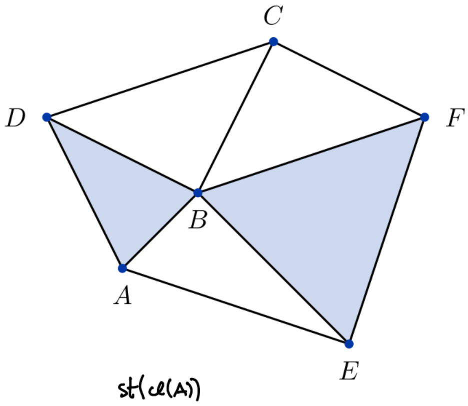
Computational topology
Lab work, 6th week

1. Find the open stars $\mathrm{st}(A)$, $\mathrm{st}(AB)$ and the links $\mathrm{lk}(A)$, $\mathrm{lk}(AB)$ for the simplicial complex given below.





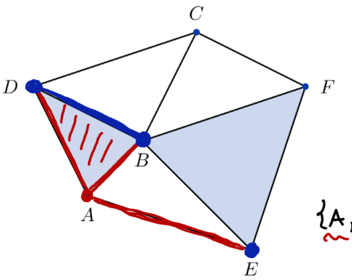
$S \ldots$ a set of simplices

$\mathrm{cl}(S)$ = a set of all simplices that are faces of simplices of S

$\mathrm{st}(\sigma)$ = set of all simplices that have σ as a face

$\mathrm{st}(S) = \bigcup_{\sigma \in S} \mathrm{st}(\sigma)$

$\mathrm{lk}(S) = \mathrm{cl}(\mathrm{st}(S)) \setminus \mathrm{st}(\mathrm{cl}(S))$



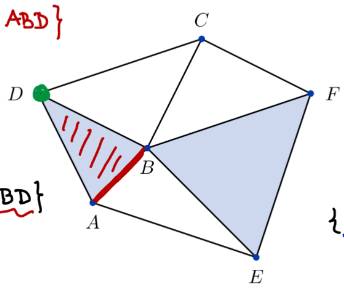
$\mathrm{st}(\mathrm{cl}(A))$ $=$ $\mathrm{st}(A) = \{A, AB, AD, AE, ABD\}$

$\mathrm{lk}(A) = \mathrm{cl}(\mathrm{st}(A)) \setminus \mathrm{st}(\mathrm{cl}(A))$

$\mathrm{cl}(A) = \{A\}$

$\{A, B, AB, D, AD, E, AE, BD, ABD\}$

$\mathrm{lk}(A) = \{B, D, E, BD\}$



$\mathrm{st}(AB) = \{AB, ABD\}$

$\mathrm{lk}(AB) = \mathrm{cl}(\mathrm{st}(AB)) \setminus \mathrm{st}(\underbrace{\mathrm{cl}(AB)}_{\{A, B, AB\}})$

$\{AB, A, B, ABD, D, AD, BD\}$

$\mathrm{st}(\mathrm{cl}(AB)) = \mathrm{st}(\{A, B, AB\}) = \mathrm{st}(A) \cup \mathrm{st}(B) \cup \mathrm{st}(AB) =$

$= \{A, AB, AD, AE, ABD, B, BE, BF, BC, BD, BEF\}$

$\mathrm{lk}(AB) = \{D\}$

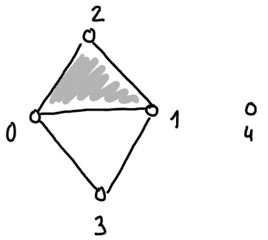
2. The simplicial complex K contains the following simplices:

$$\overset{0}{\langle v_0\rangle}, \overset{1}{\langle v_1\rangle}, \overset{2}{\langle v_2\rangle}, \langle v_3\rangle, \langle v_4\rangle, \langle v_0, v_1\rangle, \overset{03}{\langle v_0, v_3\rangle}, \langle v_1, v_3\rangle, \overset{012}{\langle v_0, v_1, v_2\rangle}.$$

(a) Add any simplices that are missing from K.

(b) Draw the Hasse diagram of K.

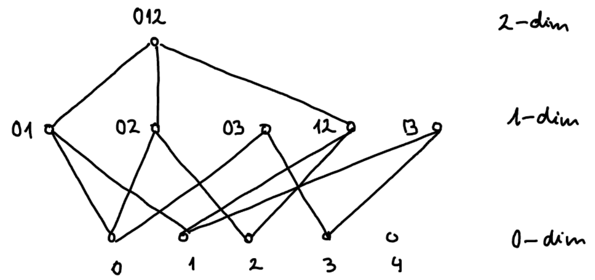
(c) Find the open stars $\mathrm{st}(\langle v_1\rangle)$, $\mathrm{st}(\langle v_1, v_3\rangle)$ and the links $\mathrm{lk}(\langle v_2\rangle)$, $\mathrm{lk}(\langle v_0, v_3\rangle)$. Mark them on the Hasse diagram as well.

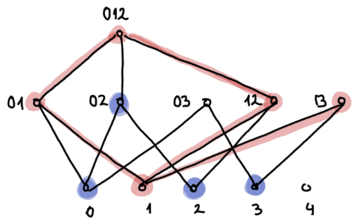


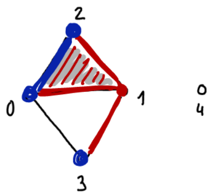
0-dim : 0, 1, 2, 3, 4

1-dim : 01, 03, 13, 02, 12

2-dim : 012







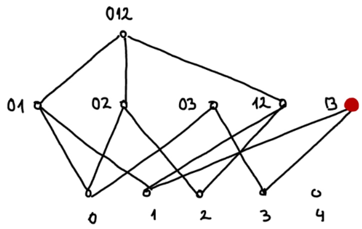
st(1) = {1, 01, 12, 13, 012} → up-set of 1 in HD

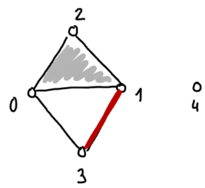
lk(1) = cl(st(1)) \ st(cl(1)) = {0, 2, 3, 02}

cl(1) = {1}

cl(st(1)) = {1, 01, 0, 12, 2, 13, 3, 012, 02}

simplices in the down-set of st(σ) that don't contain any "letters" from σ





st(13) = {13}

lk(13) = cl(st(13)) \ st(cl(13))

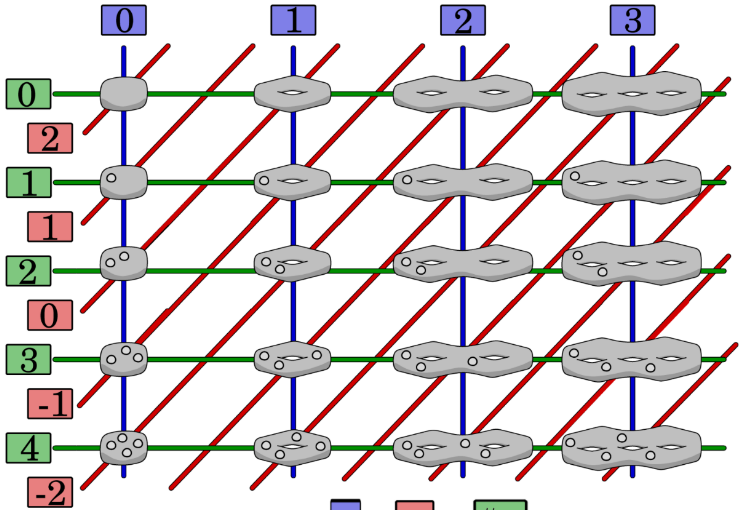
cl({13})

{1, 3, 13}

st({1, 3, 13}) = st({1}) ∪ st(3) ∪ st(13)

= {1, 01, 12, 13, 012} ∪ {3, 03, 13} ∪ {13}

lk(13) = ∅



$$2 = 2\boxed{g} + \boxed{\chi} + \boxed{\#\partial}$$

orientable

genus	0	1	2
orientable	S^2	T	$T\#T$
non-orientable		P	$P\#P$

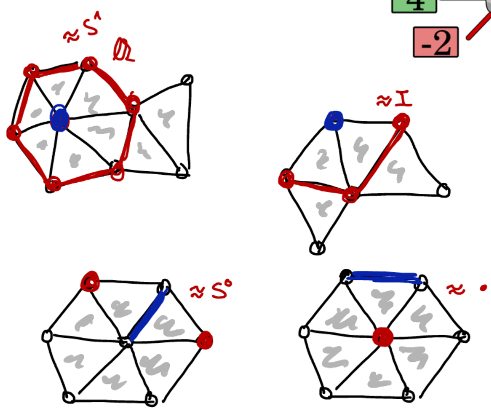
Klein bottle

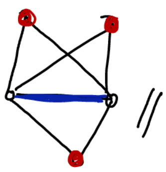
$\chi(K) = c_0 - c_1 + c_2 - \cdots + (-1)^n c_n$

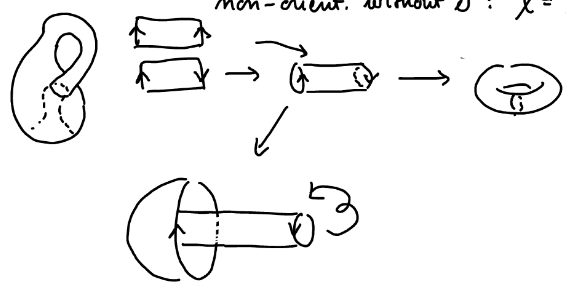
c_i = # of simplices of dimension i

orientable : $2 = 2g + \chi + \#\partial$

non-orient. without ∂ : $\chi = 2 - g$







3. For each of the following triangulations determine if it is a triangulation of a surface.

A: [(1, 2, 3), (1, 2, 4), (1, 3, 4), (2, 3, 4)]
B: [(1, 2, 3), (1, 2, 4), (2, 3, 5), (2, 3, 6), (3, 5, 7)]
C: [(1, 2, 3), (2, 3, 4), (3, 4, 5),
(4, 5, 6), (1, 5, 6), (1, 2, 6)]
D: [(1, 2, 4), (2, 4, 6), (2, 3, 6), (3, 6, 8), (1, 3, 8),
(1, 4, 8), (4, 5, 6), (5, 6, 7), (6, 7, 8), (7, 8, 9),
(4, 8, 9), (4, 5, 9), (1, 5, 7), (1, 2, 7), (2, 7, 9),
(2, 3, 9), (3, 5, 9), (1, 3, 5)]
E: [(1, 2, 4), (2, 4, 6), (2, 3, 6), (3, 6, 8), (1, 3, 8),
(1, 5, 8), (4, 5, 6), (5, 6, 7), (6, 7, 8), (7, 8, 9),
(5, 8, 9), (4, 5, 9), (1, 5, 7), (1, 2, 7), (2, 7, 9),
(2, 3, 9), (3, 4, 9), (1, 3, 4)]
F: [(1, 2, 3), (1, 3, 4), (2, 3, 4), (4, 5, 6)]
G: [(1, 2, 3), (2, 3, 4), (3, 4, 5), (4, 5, 6), (2, 5, 6), (1, 2, 6)]
H: [(1, 3, 5), (1, 2, 6), (1, 5, 6), (1, 2, 4), (1, 3, 4),
(2, 3, 5), (2, 3, 6), (2, 4, 5), (3, 4, 6), (4, 5, 6)]

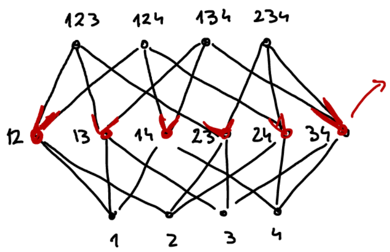
(a) Find the Euler characteristics for all of these simplicial complexes.
(b) For each case check if the given triangulation belongs to a surface (a 2-dimensional triangulated manifold).
(c) Find the number of boundary components for all of the surfaces.
(d) For each of the surfaces determine if it is orientable or not.
(e) Determine the genus of each orientable surface and the genus of non-orientable surfaces with no boundary.
(f) Name each of the surfaces.

A: [(1, 2, 3), (1, 2, 4), (1, 3, 4), (2, 3, 4)]

1, 2, 3, 4
12, 13, 23, 14, 24, 34
123, 124, 134, 234

$\chi(A) = 4 - 6 + 4 = \underline{\underline{2}}$

manifold → "HW": compute all lk of • and — → S^1 / S^0 → manifold w/o bd
→ $S^1 + I / S^0 + \bullet$ → manifold w bd

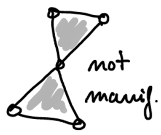


contained in exactly 2 Δ
↓
interior edge → no ∂ components → orient.?

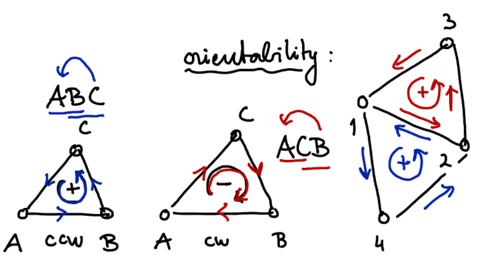
$2 = 2g + \chi + \#\partial$

$2 = 2g + 2 + 0$

$\underline{\underline{g = 0}}$ → sphere

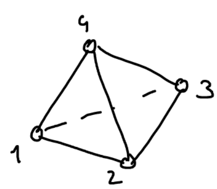


orientability:

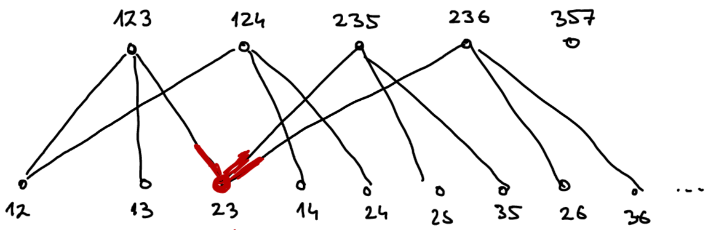


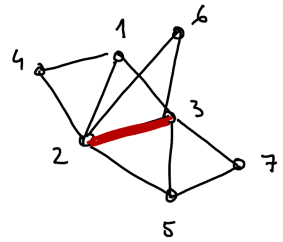
123 → 12, 23, 31
~~124 → 12, 24, 41~~
142 → 14, 42, 21 ✓
134 → 13, 34, 41 ✓
~~234 → 23, 34, 42~~
243 → 24, 43, 32 ✓

} orientable



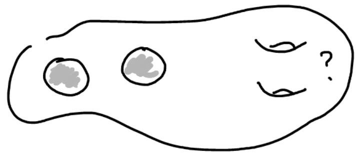
B: [(1, 2, 3), (1, 2, 4), (2, 3, 5), (2, 3, 6), (3, 5, 7)]

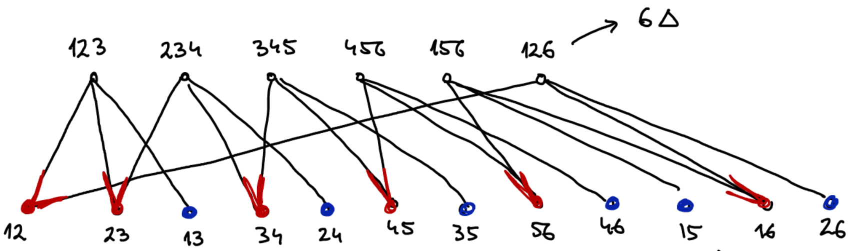




this edge is contained in 3 Δ $\to$ not a manifold

C: [(1, 2, 3), (2, 3, 4), (3, 4, 5),
(4, 5, 6), (1, 5, 6), (1, 2, 6)]

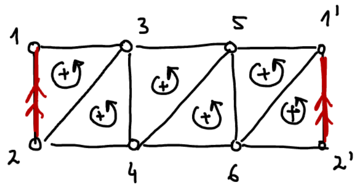




interior edges

boundary edges : {13, 24, 35, 46, 15, 26}

orientable ?



123, 243, 345, 465, 156, 162 ✓

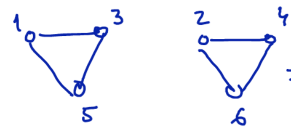
12 —

+ 6 •

$\chi(C) = 6 - 12 + 6 = 0$

$V = \{1, 3, 2, 4, 5, 6\}$

$E =$

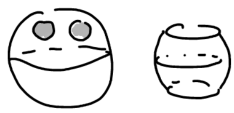


$\# \partial = 2$

$2 = 2g + \chi + \# \partial$

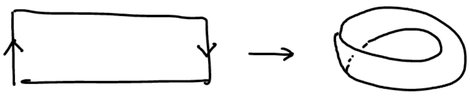
$2 = 2g + 0 + 2$

$\underline{\underline{g = 0}}$



cylinder

D: [(1, 2, 4), (2, 4, 6), (2, 3, 6), (3, 6, 8), (1, 3, 8),
(1, 4, 8), (4, 5, 6), (5, 6, 7), (6, 7, 8), (7, 8, 9),
(4, 8, 9), (4, 5, 9), (1, 5, 7), (1, 2, 7), (2, 7, 9),
(2, 3, 9), (3, 5, 9), (1, 3, 5)]



non-orientable

1 boundary component !

E: [(1, 2, 4), (2, 4, 6), (2, 3, 6), (3, 6, 8), (1, 3, 8),
(1, 5, 8), (4, 5, 6), (5, 6, 7), (6, 7, 8), (7, 8, 9),
(5, 8, 9), (4, 5, 9), (1, 5, 7), (1, 2, 7), (2, 7, 9),
(2, 3, 9), (3, 4, 9), (1, 3, 4)]

F: [(1, 2, 3), (1, 3, 4), (2, 3, 4), (4, 5, 6)]

G: [(1, 2, 3), (2, 3, 4), (3, 4, 5), (4, 5, 6), (2, 5, 6), (1, 2, 6)]

H: [(1, 3, 5), (1, 2, 6), (1, 5, 6), (1, 2, 4), (1, 3, 4),
(2, 3, 5), (2, 3, 6), (2, 4, 5), (3, 4, 6), (4, 5, 6)]

4. Let $S = \{A(0,0), B(0,1), C(0.5,0.5), D(1,2), E(1.5,1.5), F(2,0), G(2,1.5), H(2.5,1)\} \subset \mathbb{R}^2$. Build the Vietoris-Rips complex $\mathrm{Rips}(S,R)$ for

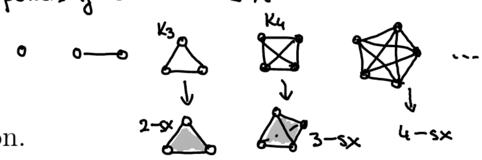
(a) $R = 1$,

(b) $R = 1.2$,

(c) $R = 1.75$.

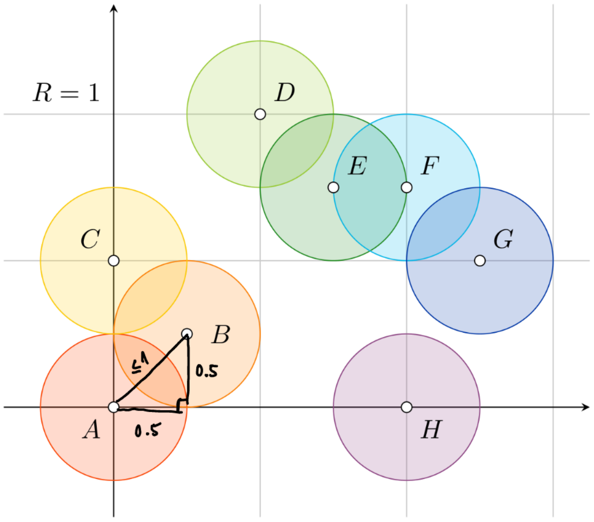
• 1-skeleton of $V(S,r)$ ... connect 2 points if distance $\leq r$

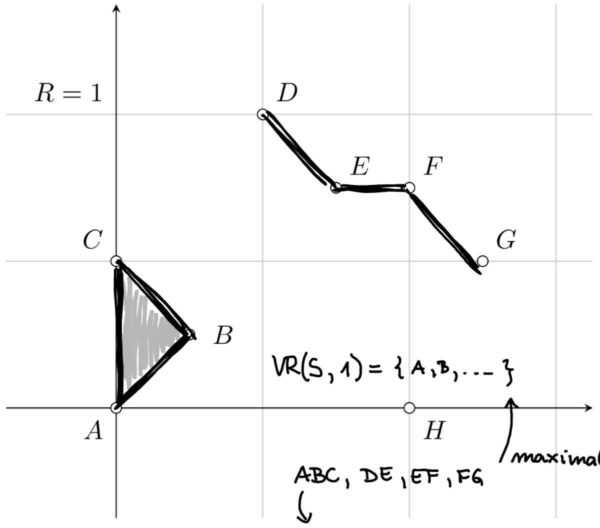
• find all cliques (complete subgraphs)



In each case list all the simplices and determine its dimension.

Assuming there is a sensor placed at each point of S and all sensors can detect points that are at distance 1.75 or less, is the area covered by the sensors connected? Does it contain any holes?



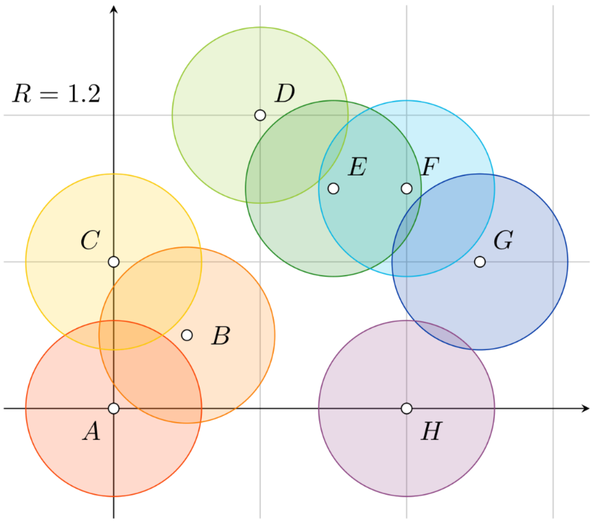


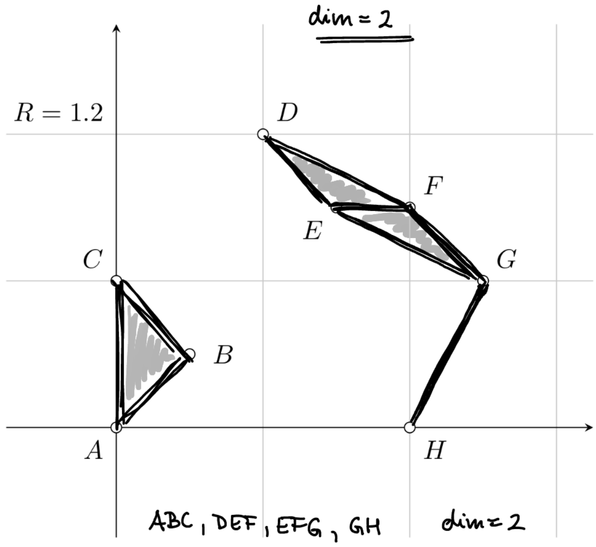
VR(S, 1) = { A, B, ... }

maximal simpliees

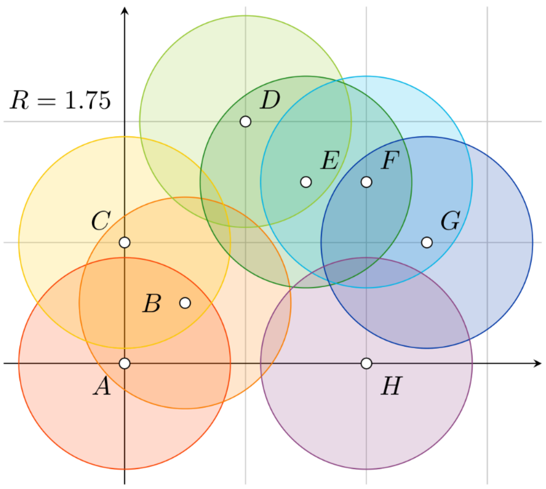
ABC, DE, EF, FG

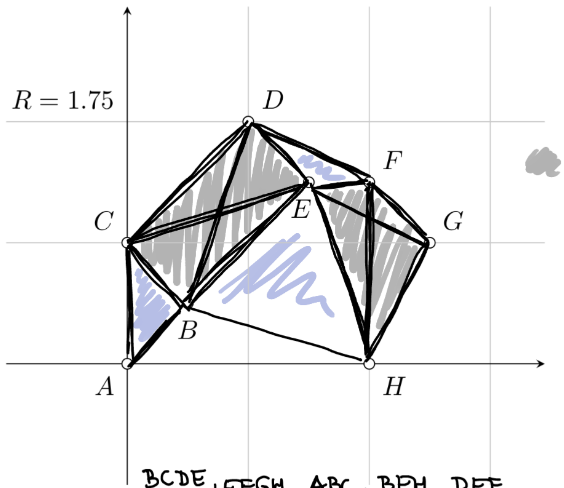
dim = 2





ABC, DEF, EFG, GH dim = 2





tetrahedron 3-sx

BCDE, EFGH, ABC, BEH, DEF

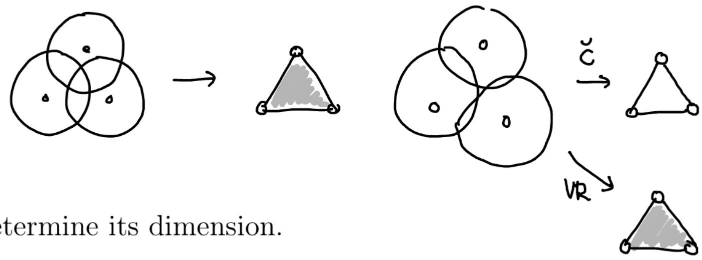
dim = 3

5. Let $S = \{A(0,0), B(0,1), C(0.5,0.5), D(1,2), E(1.5,1.5), F(2,0), G(2,1.5), H(2.5,1)\} \subset \mathbb{R}^2$. Build the Čech complex $\mathrm{Cech}(S, r)$ for

(a) $r = 0.5$,

(b) $r = 0.6$,

(c) $r = 0.875$.



In each case list all the simplices and determine its dimension.