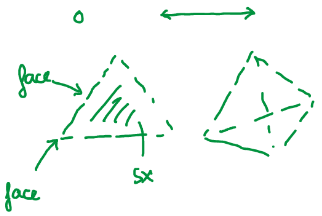
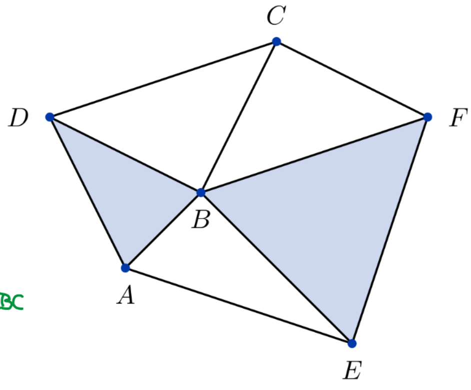
Computational topology
Lab work, 6th week

1. Find the open stars $\mathrm{st}(A)$, $\mathrm{st}(AB)$ and the links $\mathrm{lk}(A)$, $\mathrm{lk}(AB)$ for the simplicial complex given below.





S = a subset of simplices

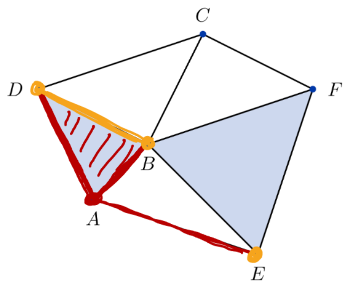
$cl(S)$ = smallest simplicial complex that contains all simplices of S

$st(S) = \bigcup_{\sigma \in S} st(\sigma)$

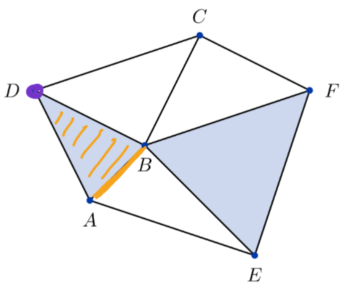
$st(\sigma)$ = set of all simplices that have σ as a face

$lk(S) = cl(st(S)) \setminus st(cl(S))$

$\underbrace{ABC}_{\sigma} \to \underbrace{A, B, C, AB, AC, BC, ABC}_{cl(\sigma)}$



$st(A) = \{A, AD, AB, AE, ABD\}$



$st(AB) = \{AB, ABD\}$

$lk(A) = cl(st(A)) \setminus st(\overbrace{cl(A)}^{\{A\}}) = \{D, B, E, BD\}$

$\{A, D, B, E, AD, AB, AE, BD, ABD\}$

$lk(AB) = cl(st(AB)) \setminus st(cl(AB))$

$st(\{A, B, AB\}) = st(A) \cup st(B) \cup st(AB)$

$\{A, B, D, AB, AD, BD, ABD\}$

$\{A, AD, AB, AE, ABD, B, BE, BF, BC, BD, BEF\}$

$lk(AB) = \{D\}$

2. The simplicial complex K contains the following simplices:

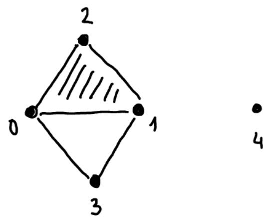
$$\underset{}{\overset{0}{\langle v_0\rangle}}, \overset{1}{\langle v_1\rangle}, \langle v_2\rangle, \langle v_3\rangle, \langle v_4\rangle, \overset{01}{\langle v_0, v_1\rangle}, \overset{03}{\langle v_0, v_3\rangle}, \overset{13}{\langle v_1, v_3\rangle}, \overset{012}{\langle v_0, v_1, v_2\rangle}. \qquad \boxed{12}\ \boxed{02}$$

(a) Add any simplices that are missing from K.

(b) Draw the Hasse diagram of K.

(c) Find the open stars $\mathrm{st}(\langle v_1\rangle)$, $\mathrm{st}(\langle v_1, v_3\rangle)$ and the links $\mathrm{lk}(\langle v_2\rangle)$, $\mathrm{lk}(\langle v_0, v_3\rangle)$. Mark them on the Hasse diagram as well.

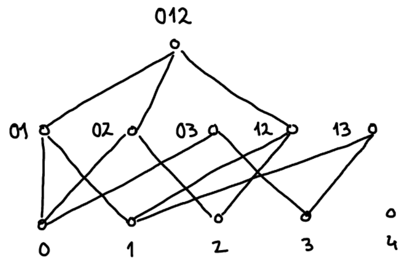
a)



0, 1, 2, 3, 4

01, 02, 03, 12, 13

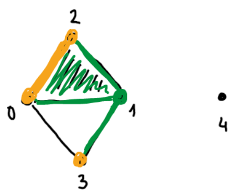
012

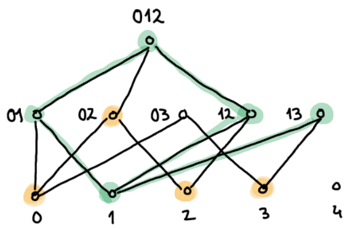


dim = 2

dim = 1

dim = 0

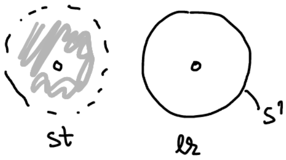




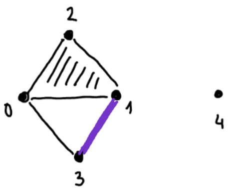
$\mathrm{st}(1) = \{1, 01, 12, 13, 012\}$ → up-set of $\{1\}$

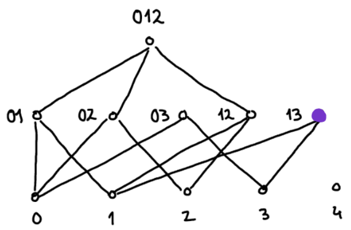
$\mathrm{lk}(1) = \mathrm{cl}(\mathrm{st}(1)) \setminus \mathrm{st}(\underbrace{\mathrm{cl}(1)}_{\{1\}})$

$\{1, 0, 2, 3, 01, 12, 13, 02, 012\}$



$\mathrm{lk}(1) = \{0, 2, 3, 02\}$ → down-set of $\mathrm{st}(1)$, but without $\mathrm{st}(1)$, without everything "intersecting" 1





$\mathrm{st}(13) = \{13\}$

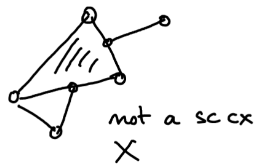
$\mathrm{lk}(13) = \mathrm{cl}(\mathrm{st}(13)) \setminus \mathrm{st}(\mathrm{cl}(13))$

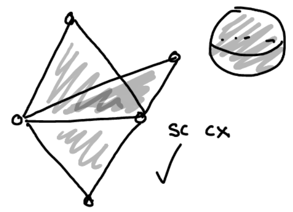
$\{1, 3, 13\}$

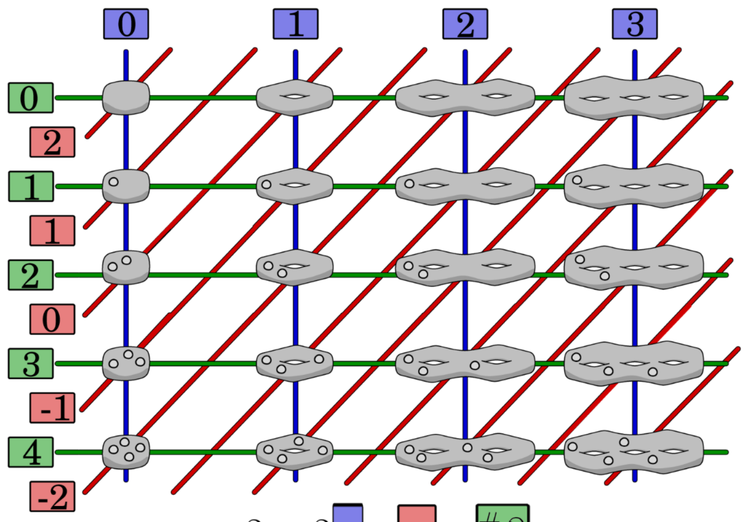
$\mathrm{st}(\{1, 3, 13\}) = \mathrm{st}(1) \cup \mathrm{st}(3) \cup \mathrm{st}(13) =$

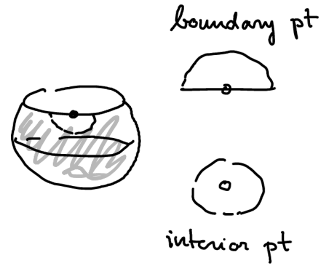
$= \{1, 01, 12, 13, 012, 3, 03, 13\}$

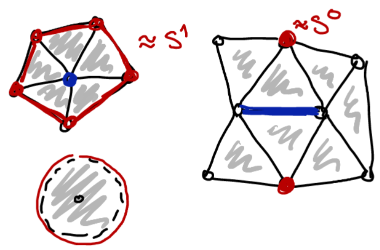
$\mathrm{lk}(13) = \emptyset$











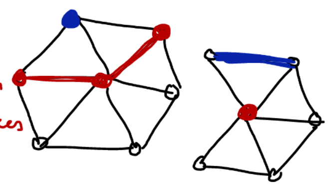
$$2 = 2\boxed{g} + \boxed{\chi} + \boxed{\#\partial}$$

genus

$\chi(K) = c_0 - c_1 + c_2 - c_3 + \cdots + (-1)^n c_n$

c_i = # of simplices of dim i

genus	0	1	2
orientable	S^2	T	$T\#T$
non-orientable		P	$P\#P$



A 2-dim manifold without boundary is a 2-dim sc cx s.t.

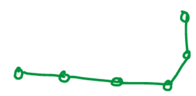
- $lk(v)$ is $\approx S^1$
- $lk(uv)$ is $\approx S^0$

A .... with boundary is ...

- $lk(v)$ is $\approx S^1$ or a path $\approx I$
- $lk(uv)$ is $\approx S^0$ or a single point

interior

boundary edges/points



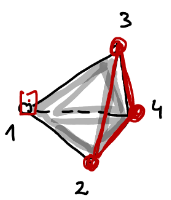
$I = [0,1]$

3. For each of the following triangulations determine if it is a triangulation of a surface.

A: [(1, 2, 3), (1, 2, 4), (1, 3, 4), (2, 3, 4)]
B: [(1, 2, 3), (1, 2, 4), (2, 3, 5), (2, 3, 6), (3, 5, 7)]
C: [(1, 2, 3), (2, 3, 4), (3, 4, 5),
(4, 5, 6), (1, 5, 6), (1, 2, 6)]
D: [(1, 2, 4), (2, 4, 6), (2, 3, 6), (3, 6, 8), (1, 3, 8),
(1, 4, 8), (4, 5, 6), (5, 6, 7), (6, 7, 8), (7, 8, 9),
(4, 8, 9), (4, 5, 9), (1, 5, 7), (1, 2, 7), (2, 7, 9),
(2, 3, 9), (3, 5, 9), (1, 3, 5)]
E: [(1, 2, 4), (2, 4, 6), (2, 3, 6), (3, 6, 8), (1, 3, 8),
(1, 5, 8), (4, 5, 6), (5, 6, 7), (6, 7, 8), (7, 8, 9),
(5, 8, 9), (4, 5, 9), (1, 5, 7), (1, 2, 7), (2, 7, 9),
(2, 3, 9), (3, 4, 9), (1, 3, 4)]
F: [(1, 2, 3), (1, 3, 4), (2, 3, 4), (4, 5, 6)]
G: [(1, 2, 3), (2, 3, 4), (3, 4, 5), (4, 5, 6), (2, 5, 6), (1, 2, 6)]
H: [(1, 3, 5), (1, 2, 6), (1, 5, 6), (1, 2, 4), (1, 3, 4),
(2, 3, 5), (2, 3, 6), (2, 4, 5), (3, 4, 6), (4, 5, 6)]

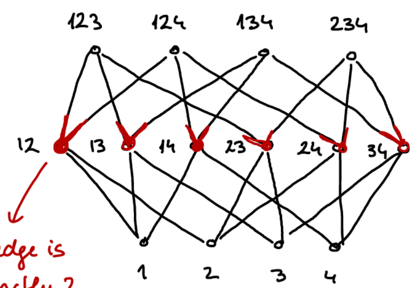
(a) Find the Euler characteristics for all of these simplicial complexes.
(b) For each case check if the given triangulation belongs to a surface (a 2-dimensional triangulated manifold).
(c) Find the number of boundary components for all of the surfaces.
(d) For each of the surfaces determine if it is orientable or not.
(e) Determine the genus of each orientable surface and the genus of non-orientable surfaces with no boundary.
(f) Name each of the surfaces.

A: [(1, 2, 3), (1, 2, 4), (1, 3, 4), (2, 3, 4)]



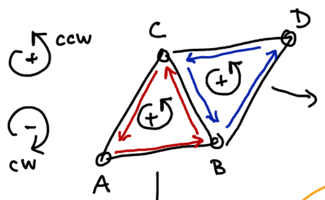
0-dim : 1, 2, 3, 4
1-dim : 12, 13, 23, 14, 24, 34
2-dim : 123, 124, 134, 234

$\chi(A) = 4 - 6 + 4 = \underline{\underline{2}}$
$\chi = 2$



this edge is in exactly 2 triangles
interior edge → no boundary edges
$\# \partial = 0$

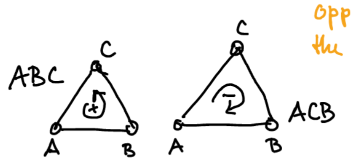
Orientable ?



BDC → BD, DC, CB

ABC → AB, BC, CA

consistent orientation on △ = opposite orientation on the edges



$2 = 2g + \chi + \#\partial$
$2 = 2g + 2 + 0$
$\boxed{g=0}$ → genus 0, no boundary, manifold
orientable ✓

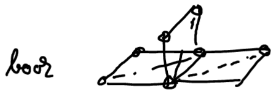
"HW": check lk of edges and vertices

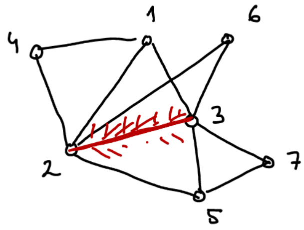
123 → 12, 23, 31 ✓
124 → 12, 24, 41 //
134 → 13, 34, 41 ✓

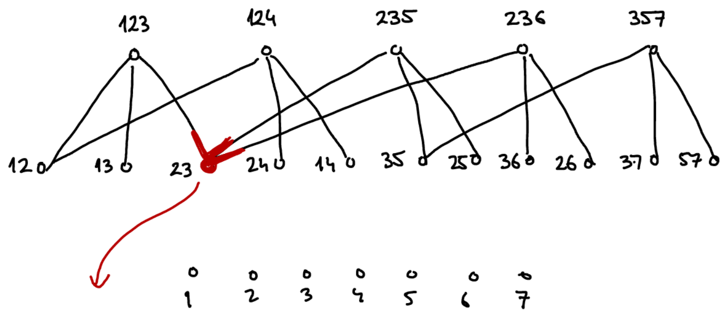
{123, 142, 134, 243} oriented
142 → 14, 42, 21 ✓
234 → 23, 34, 42 //
243 → 24, 43, 32 ✓

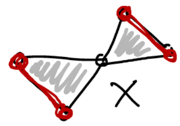
⇒ A = sphere

B: [(1, 2, 3), (1, 2, 4), (2, 3, 5), (2, 3, 6), (3, 5, 7)]









not a manifold

C: [(1, 2, 3), (2, 3, 4), (3, 4, 5),
(4, 5, 6), (1, 5, 6), (1, 2, 6)]

D: [(1, 2, 4), (2, 4, 6), (2, 3, 6), (3, 6, 8), (1, 3, 8),
(1, 4, 8), (4, 5, 6), (5, 6, 7), (6, 7, 8), (7, 8, 9),
(4, 8, 9), (4, 5, 9), (1, 5, 7), (1, 2, 7), (2, 7, 9),
(2, 3, 9), (3, 5, 9), (1, 3, 5)]

E: [(1, 2, 4), (2, 4, 6), (2, 3, 6), (3, 6, 8), (1, 3, 8),
(1, 5, 8), (4, 5, 6), (5, 6, 7), (6, 7, 8), (7, 8, 9),
(5, 8, 9), (4, 5, 9), (1, 5, 7), (1, 2, 7), (2, 7, 9),
(2, 3, 9), (3, 4, 9), (1, 3, 4)]

F: [(1, 2, 3), (1, 3, 4), (2, 3, 4), (4, 5, 6)]

G: [(1, 2, 3), (2, 3, 4), (3, 4, 5), (4, 5, 6), (2, 5, 6), (1, 2, 6)]

H: [(1, 3, 5), (1, 2, 6), (1, 5, 6), (1, 2, 4), (1, 3, 4),
(2, 3, 5), (2, 3, 6), (2, 4, 5), (3, 4, 6), (4, 5, 6)]

4. Let $S = \{A(0,0), C(0,1), B(0.5,0.5), D(1,2), E(1.5,1.5), F(2,0), G(2,1.5), H(2.5,1)\} \subset \mathbb{R}^2$. Build the Vietoris-Rips complex Rips(S,R) for

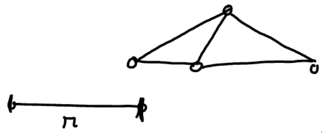
$V(S, r)$

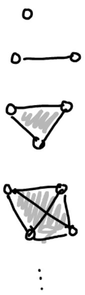
- 1-skeleton of $V(S,r)$ = connect all the points at distance $\leq r$
- add all cliques

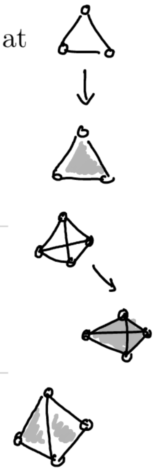
(a) $R = 1$,

(b) $R = 1.2$,

(c) $R = 1.75$.

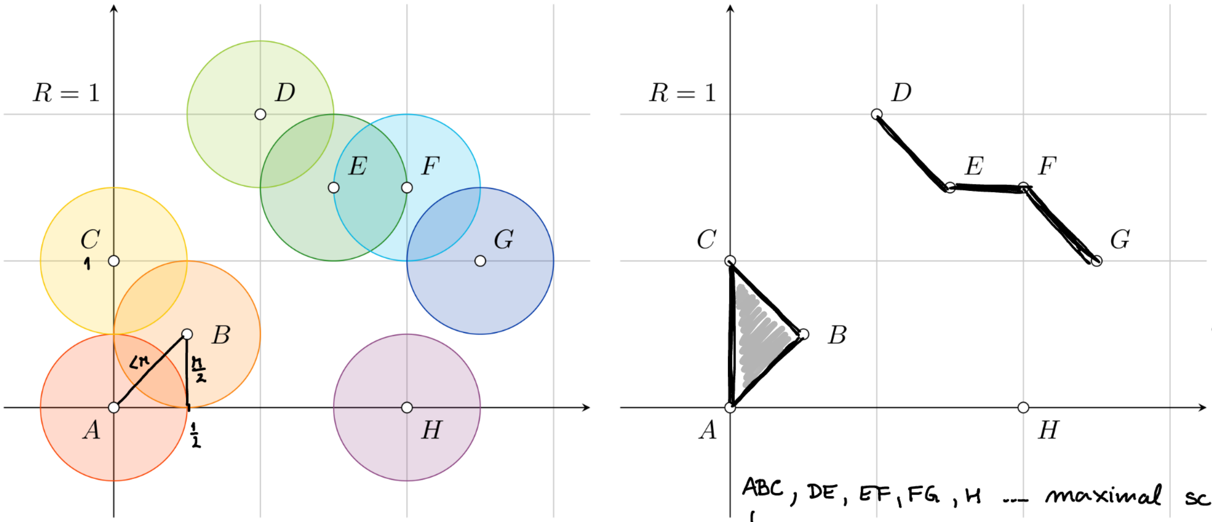






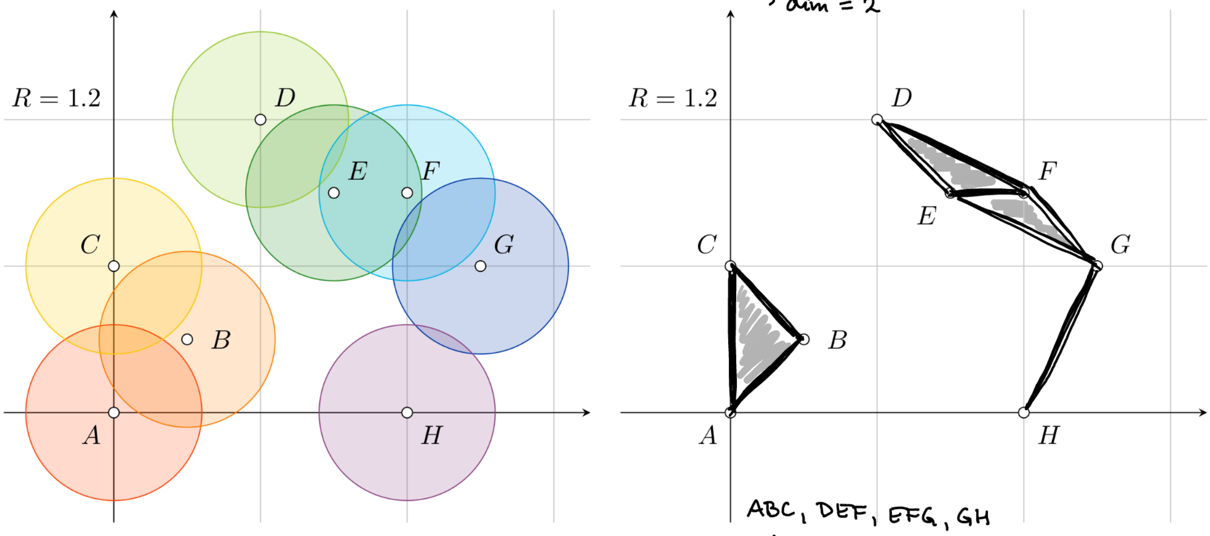
In each case list all the simplices and determine its dimension.

Assuming there is a sensor placed at each point of S and all sensors can detect points that are at distance 1.75 or less, is the area covered by the sensors connected? Does it contain any holes?



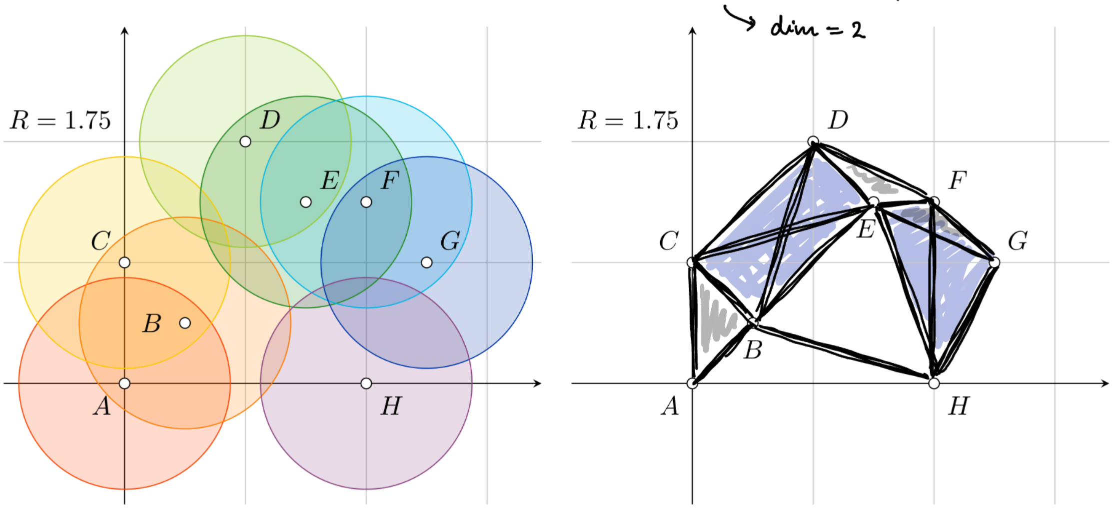
ABC, DE, EF, FG, H --- maximal sc of $V(S,1)$

dim = 2



ABC, DEF, EFG, GH

dim = 2



BCDE, EFGH, DF, BH, ABC

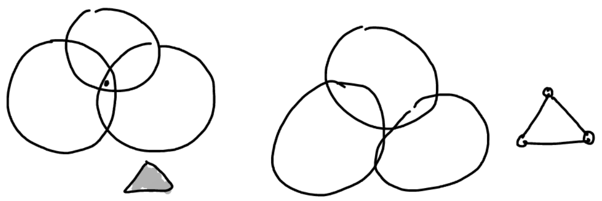
dim = 3

5. Let $S = \{A(0,0), B(0,1), C(0.5,0.5), D(1,2), E(1.5,1.5), F(2,0), G(2,1.5), H(2.5,1)\} \subset \mathbb{R}^2$. Build the Čech complex $\mathrm{Cech}(S, r)$ for

(a) $r = 0.5$,

(b) $r = 0.6$,

(c) $r = 0.875$.



In each case list all the simplices and determine its dimension.