Homology: Impact and computation by parts

Žiga Virk

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Homology as defined in the previous chapter is an invariant assigned to a simplicial complex. While its homotopy invariance and computational amenability make homology a suitable tool for computational purposes, the structural depth of the underlying theory goes far beyond the presented material.

In this chapter we present some further properties on homology. The first one is functoriality and its impact on significant topological results of the beginning of the twentieth century. The second one is the ability to combine homology computations of two parts of a space in order to deduce the homology of the whole space.

1 Impact

One of the fundamental tasks of mathematics is a construction of new objects (invariants) assigned to known objects. For example, given a closed surface we can assign to it a triangulation. In turn, we can assign homology groups to that obtained triangulation.

It turns out to be supremely beneficial if such an assignment can be extended in a consistent manner to maps between the original objects as well. When this is the case we say the assignment is functorial[1]. It turns out that homology is functorial as Proposition 1.2 demonstrates.

[1] Functoriality and its formal consequences are studied within the Category Theory.

Functoriality of homology

Definition 1.1. *Suppose $f\colon K \to L$ is a simplicial map between simplicial complexes, $q \in \{0, 1, \dots\}$, and $\mathbb{F}$ is a field. The* ***induced maps*** *$f_\#$ and f_* are defined as follows:*

- *$f_\#\colon C_q(K;\mathbb{F}) \to C_q(L;\mathbb{F})$ is the linear map of chain groups defined as*

$$f_\#\Big(\sum_i a_i\sigma_i\Big) = \sum_{i\,:\,\dim(f(\sigma_i))=q} a_i f(\sigma_i), \qquad a_i \in \mathbb{F}, \sigma_i^{(q)} \in K.$$

- *$f_*\colon H_q(K;\mathbb{F}) \to H_q(L;\mathbb{F})$ is the linear map defined as*

$$f_*([\alpha]) = [f_\#(\alpha)].$$

⚠ We refrain from specifying q and $\mathbb{F}$ in the notation f_* in order not to overload it with the indices. As such f_* represents the induced map on homology in any dimension or coefficients. The relevant choice of the dimension(s) and coefficients should always be apparent from the context.

☞ The induced maps in the case of coefficients in a group are homomorphisms and are still well defined.

☞ Identity maps between spaces induce identity maps on homology. Constant maps between spaces induce trivial (i.e., zero) maps on homology.

Comments on Definition 1.1 using the notation established in it:

1. Given a simplex $\sigma \in K$ of dimension q, its image $f(\sigma)$ is a simplex of dimension q or less. The condition on dimension in the definition

of $f_\#$ means that only the images of those simplices σ_i, which are of full dimension q, are taken into account. In particular,

$$f_\#(\sigma) = \begin{cases} f(\sigma); & \dim(f(\sigma)) = q \\ 0; & \text{else.} \end{cases}$$

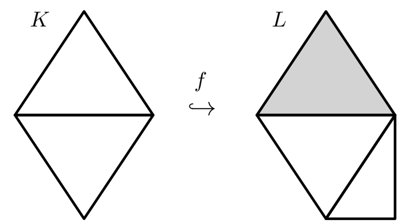

Figure 1: An embedding $f\colon K \to L$. While the first homology groups of K and L are of dimension 2, the image $f_*(H_1(K;\mathbb{F}))$ is of dimension 1, demonstrating that the embedding preserves only one hole. This interpretation will be significantly expanded within the context of persistent homology.

2. The induced map f_* turns out to be well defined, i.e., if $[\alpha] = [\beta]$ then $f_*([\alpha]) = f_*([\beta])$.
3. Homotopic maps induce the same maps on homology.
4. Suppose X and Y are metric spaces with triangulations K and L. By the Simplicial approximation Theorem there exists for each continuous map $f_1\colon X \to Y$ a simplicial map f_2 between some subdivisions of K and L, such that $f_1 \simeq f_2$. Whenever we mention the homology of X we formally think of the homology of K. In a similar manner, whenever we talk about the maps on homology induced by a continuous map f_1, we formally think[2] of maps induced by the simplicial map f_2.

[2] With this explanation, the notion of a map on homology induced by a continuous map between spaces is well defined.

The induced maps are consistent with respect to compositions[3] as the following proposition explains.

[3] Formally speaking we express this property by saying that homology is functorial.

Proposition 1.2. *[Functoriality of the induced maps] Suppose maps $f\colon K \to L$ and $g\colon L \to M$ between simplicial complexes are simplicial. Then for each $q \in \{0,1,\dots\}$ and for each $\mathbb{F}$ we have*

$$(g\circ f)_\# = g_\# \circ f_\#, \qquad \text{and} \qquad (g\circ f)_* = g_* \circ f_*.$$

The proof follows straight from the definition.

One of the most natural demonstrations of the power of functoriality concerns the existence of retractions. Given a space X and its subspace[4] $A \subset X$, a **retraction** of X to A is any continuous map $X \to A$ such that $f(a) = a, \forall a \in A$.

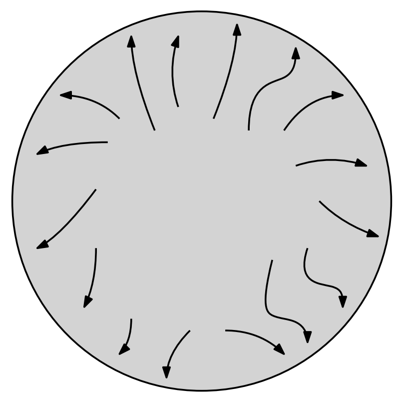

Figure 2: A geometric intuition dictates that if we want to retract B^2 onto $S^1 = \partial B^2$, the resulting map would need to have a discontinuity, i.e., at least one point where we "tear" the disc. A fairly simple proof of this fact is given using homology.

[4] A required condition for the existence of a retraction is for A to be closed in X.

Example 1.3. *For each $n \in \mathbb{N}$ the standard $(n-1)$-sphere S^{n-1} is the boundary of the standard n-ball B^n. We claim there is no retraction $B^n \to S^{n-1}$. As a special case, there is no retraction of the unit interval onto its endpoints.*

Proof. Assume such a retraction $f\colon B^n \to S^{n-1} = \partial B^n$ exists. Precompose it with the inclusion $g\colon S^{n-1} \hookrightarrow B^n$, see Figure 3. Let $[\alpha] \neq 0$ be a basis (generator) of $H_{n-1}(S^{n-1};\mathbb{F})$. We combine two observations:

- As $f \circ g\colon S^{n-1} \to S^{n-1}$ is identity, $(f\circ g)_*([\alpha]) = [\alpha] \neq 0$.
- As $H_{n-1}(B^n;\mathbb{F}) = 0$, $g_*([\alpha]) = 0$ and thus $f_*(g_*([\alpha])) = 0$.

By Proposition 1.2 $(f\circ g)_*([\alpha]) = f_*(g_*([\alpha]))$, a contradiction. Hence a retraction f does not exist. □

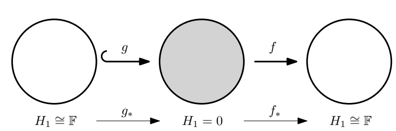

Figure 3: The proof of Example 1.3. The composition of maps is identity on S^1, while the composition of induced maps can't be identity as it factors through 0.

Brouwer fixed point

Brouwer fixed point Theorem is probably one of the most famous early results of topology. It has a surprisingly short proof using the functoriality of homology.

Theorem 1.4. *Every continuous map $f\colon B^2 \to B^2$ has a fixed point, i.e., a point $x_0 \in B^2$ such that $f(x_0) = x_0$.*

Proof. Assume map f has no fixed point. Define map $g\colon B^2 \to S^1$ by declaring that for each $x \in B^2$, point $g(x) \in S^1 = \partial B^2$ is the intersection of S^1 with the ray based at $f(x)$ containing x, see Figure 4. As f has not fixed point, the mentioned ray is always unique. Map g is a continuous retraction, a contradiction according to Example 1.3. Hence a fixed point exists. □

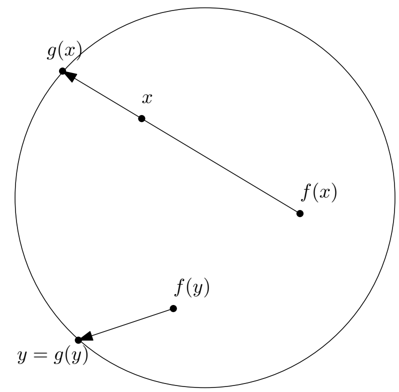

Figure 4: Map g from the proof of the Brouwer fixed point Theorem.

Hairy ball

Another prominent theorem that can be conveniently proved using homology is the Hairy ball Theorem. The name comes from a popular adaptation of the result: one can't comb a hair on a hairy ball without creating a hair whorl.

Before we state the theorem we need to clarify a few technical details.

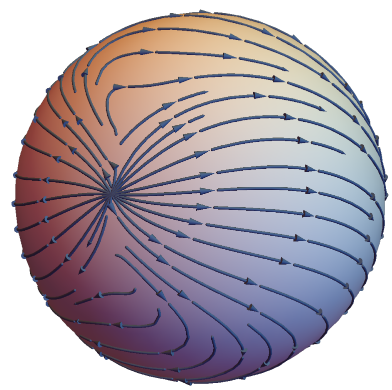

Figure 5: A tangent vector field on a sphere induces a flow presented by the streamlines on this figure. Theorem 1.5 states that the vector field must have a zero, which can be demonstrated on our example by the source of streamlines. To the contrary, there are non-trivial tangent vector fields in the plane and on the torus.

1. A **tangent vector field** on a surface X is a continuous map $f\colon X \to \mathbb{R}^3$ such that for each $x \in S$ we have $x \perp f(x)$. A vector field f is non-vanishing, if it is non-zero at each point.

2. Given a centrally symmetric[5] triangulation K of S^2, let $\alpha = \sum_{\sigma^{(2)} \in K} \sigma$ be the cycle defined as the sum of consistently oriented triangles of K. Without the loss of generality we may assume the triangles are oriented so that their "upwards" direction is pointing away from the point $(0,0,0)$. Recall that $[\alpha]$ is the fundamental class spanning $H_2(K;\mathbb{R}) \cong \mathbb{R}$.

3. For each triangle $\sigma \in K$ the reflection of σ through $(0,0,0)$ is again a simplex σ' of K. However, if σ has the chosen orientation from the previous point[6], then the reflected triangle has the opposite orientation[7] from the originally chosen orientation on σ', see the left portion of Figure 6. In particular, $[\alpha'] = \sum_{\sigma^{(2)} \in K} \sigma'$ is a non-trivial homology class representing $-[\alpha]$.

4. Let $\rho\colon K \to K$ be the reflection map and let $g\colon K \to K$ be the identity map. By 2. and 3. maps g and ρ are not homotopic as $g_*([\alpha]) = [\alpha] \neq \rho_*([\alpha]) = [\alpha']$.

[5] I.e., the triangulation K has the following property: for each simplex $\tau \in K$ its reflection through the point $(0,0,0)$ is also a simplex.

[6] I.e., such that the chosen normal is pointing away from the point $(0,0,0)$.

[7] I.e., such that the chosen normal is pointing towards the point $(0,0,0)$.

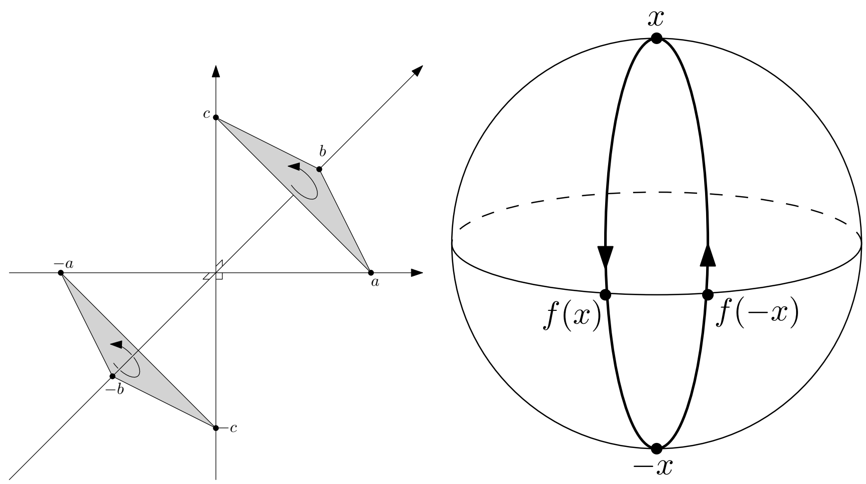

Figure 6: Elements of the proof of Theorem 1.5.

On the left side is simplex $\langle a, b, c\rangle$ and its (oriented) reflection through $(0, 0, 0)$: $\langle -a, -b, -c\rangle$. Observe that in both cases the normal to the simplex is in the direction $(1, 1, 1)$. The same argument and picture work for any odd dimension, which leads to Theorem 1.6.

On the right side is the construction of homotopy from the proof of Theorem 1.5. Point x is connected to $-x$ by the geodesic passing through $f(x)$ and vice versa.

Theorem 1.5. *Every tangent vector field on S^2 has a zero.*

Proof. Suppose f is a non-vanishing vector field on S^2. Without the loss of generality[8] we can assume $||f(x)|| = 1, \forall x \in S^2$. Using the notation of the discussion leading to this theorem, we will prove that $g \simeq \rho$, which is a contradiction by 4. of the mentioned discussion.

We will construct an explicit homotopy between g and ρ. Such a homotopy can be thought of as a continuous collection[9] of paths from x to $-x$ for all[10] $x \in S^2$.

The simplest way to connect two diametrally opposite points on a sphere, for the sake of simplicity let us assume we are connecting the north pole N to the south pole S, is by drawing a meridian between them. Such a meridian is completely determined by the point at which it intersects the equator. We define this intersection point to be[11] $f(N)$.

In general, connect x to $-x$ by a geodesic[12] on the sphere passing through $f(x)$. This is a continuous assignment of paths and constitutes the homology between g and ρ, which completes the proof. $\square$

[8] I.e., by normalizing each vector in the image of f.

[9] A homotopy in question is of the form $H \colon S^2 \times [0, 1] \to S^2$. For each $y \in S^2$ the restriction $H|_{\{y\}\times[0,1]}$ is thus a path from y to $-y$. The fact that H is continuous means that the collection of mentioned paths is continuous.

[10] While the homology setup above is performed in the simplicial setting of K, the homology here will be constructed on a “smooth” sphere S^2.

[11] Recall that $||f(N)|| = 1$ and $f(N) \perp N$, hence $f(N)$ lies on the equator.

[12] This geodesic traces the trail of x as translated by the resulting homotopy. On the other hand, the trail of $-x$ as translated by the resulting homotopy is given by the geodesic from $-x$ to x passing through $f(-x)$. See the right portion of Figure 6 for a sketch.

The argument of Theorem 1.5 works for any even dimension which leads to a more general result.

Theorem 1.6. *Sphere S^n admits a non-vanishing tangent field iff n is even.*

When n is even there is an easy construction of a non-vanishing tangent field:

$$(x_1, y_1, x_2, y_2, \ldots, x_m, y_m) \mapsto (y_1, -x_1, y_2, -x_2, \ldots, y_m, -x_m).$$

Invariance of domain

The last classical result we mention explains why open sets in Euclidean spaces of different dimension are fundamentally different in the sense that they can't be homeomorphic[13]. We will actually only prove a corollary of the classical result[14].

[13] While homology itself is a homotopy invariant, the trick we will use will allow us to use it to differentiate homeomorphic types of spaces.

[14] Effectively we will prove that $D^n \not\cong D^m$ if $m \neq n$.

> **Theorem 1.7.** *For any pair of natural numbers $m \neq n$ the closed balls $B_1 = B_{\mathbb{R}^m}(0,1)$ and $B_2 = B_{\mathbb{R}^n}(0,1)$ are not homeomorphic.*

Proof. Assume there exists a homeomorphism $f\colon B_1 \to B_2$. Let $x \in B_1$ be the center of B_1. Then $f|_{B_1\setminus\{x_1\}}\colon B_1 \setminus \{x_1\} \to B_2 \setminus \{f(x_1)\}$ is also a homeomorphism. Recall that $B_1 \setminus \{x_1\} \simeq S^{n-1}$ via the radial projection (see Figure 7), which means $H_{n-1}(B_1 \setminus \{x_1\}; \mathbb{F})$ is nontrivial for any $\mathbb{F}$. On the other hand, $B_2 \setminus \{f(x_1)\}$ is either:

- homotopy equivalent to S^{m-1} if $f(x_1) \notin \partial B_2$, or
- contractible if $f(x_1) \in \partial B_2$.

In both cases $H_{n-1}(B_2 \setminus \{f(x_1)\}; \mathbb{F}) = 0$, a contradiction, hence f can't exist. □

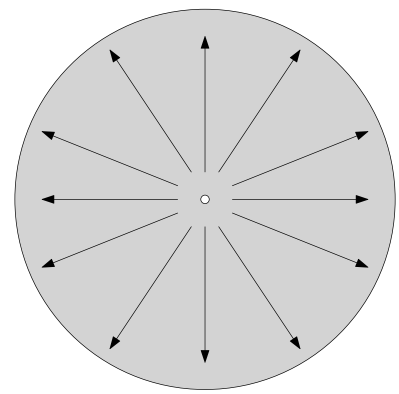

Figure 7: Radial projection of a disc with the center removed to the boundary of the disc. The induced homotopy equivalence demonstrates $B_1 \setminus \{x_1\} \simeq S^{n-1}$ in the proof of Theorem 1.7.

2 Homology by parts

Given a decomposition of a simplicial complex $K = A \cup B$ as the union of subcomplexes A and B, can[15] we compute the homology of X from the homology of A and B?

[15] A similar question: Given a finite set $Y = C \cup D$, can we determine the cardinality $|Y|$ from $|C|$ and $|D|$? The answer $|Y| = |C| + |D| - |C \cap D|$, which also includes the intersection, is not unlike the answer to our question about homology...especially since, for discrete sets, the cardinality represents the zero-dimensional homology.

The answer to this question is unfortunately negative, for example:

- As the sidenote on the right on a similar question demonstrates, the cumulative zero-dimensional homology of K and L may be too large and should possibly be decreased by the zero-dimensional union of the intersection.
- On the other hand, a circle is the union of two intervals, i.e., a space with a one-dimensional hole is the union of two subspaces without holes.

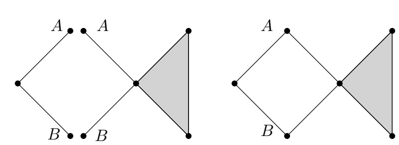

Figure 8: Two contractible complexes, whose union is not contractible.

These two examples show that A and B can have cumulatively "too much" or "too little" homology to deduce the homology of the union X and that one should probably take into account the homology of the intersections as well. The algebraic structure through which the connection between the homologies of X, A, and B is expressed is that of exact sequences.

Exact sequences

Definition 2.1. *A sequence of vector spaces $V_0, V_1, \ldots$ and linear maps $\varphi_n \colon V_n \to V_{n-1}$ is* ***exact****, if for each n we have $\operatorname{Im}\varphi_{n+1} = \ker f_n$.*

☞ Recall that homology is defined from a sequence of chain groups called the chain complex; it is defined as the quotient $\ker\partial / \operatorname{Im}\partial$. In particular, the homology of a chain complex is zero at all dimensions iff the chain complex forms an exact sequence. Or, to put it locally, $H_q = 0$ iff the chain complex is "exact at C_q" in the sense that $\operatorname{Im}\partial_{q+1} = \ker\partial_q$. Homology thus measures the extent to which a chain complex is not exact.

It turns out that in an exact sequence, the dimension of each vector space (except for the last one) can be deduced from the ranks of the neighboring maps.

Proposition 2.2. *Suppose the following sequence is exact:*

$$\cdots \to V_{n+1} \xrightarrow{\varphi_{n+1}} V_n \xrightarrow{\varphi_n} V_{n-1} \to \cdots \to V_2 \xrightarrow{\varphi_2} V_1 \xrightarrow{\varphi_1} V_0.$$

Then for each $n > 0$, $\dim V_n = \operatorname{rank}\varphi_{n+1} + \operatorname{rank}\varphi_n$.

Proof. We know that $\dim V_n = \dim\ker\varphi_n + \operatorname{rank}\varphi_n$. Now use exactness: $\operatorname{Im}\varphi_{n+1} = \ker\varphi_n$. □

Mayer-Vietoris exact sequence

We are now able to express[16] the connection between the homology of X and the homology of its two parts A and B, a connection that also includes the homology of the intersection $A \cap B$.

[16] Standard proofs use Zig-Zag lemma given in an appendix.

Theorem 2.3. *Suppose $A, B \leq K$ are subcomplexes of a simplicial complex K such that $A \cup B = X$. Then for each choice of coefficients the following sequence of homology groups is exact:*

$$\cdots \to H_{n+1}(X) \xrightarrow{\delta_{n+1}} H_n(A \cap B) \xrightarrow{(i_n, j_n)} H_n(A) \oplus H_n(B) \xrightarrow{\mu_n} H_n(X) \to \cdots$$

$$\cdots \to H_0(A \cap B) \xrightarrow{(i_0, j_0)} H_0(A) \oplus H_0(B) \xrightarrow{\mu_0} H_0(X) \to 0,$$

with the involved maps being defined as follows:

- *i_*, j_* are inclusion induced maps, i.e., $i_*[\alpha] = [\alpha]$ and $j_*[\alpha] = [\alpha]$.*
- *μ is the subtraction map, i.e., $mu_*([\alpha], [\beta]) = [\alpha - \beta]$.*
- *δ is a variant of a boundary map defined as follows. Given an n-cycle α in X, decompose it as $\alpha = \alpha_A + \alpha_B$ where α_A is an n-chain in A and α_B is an n-chain in B. Define $\delta[\alpha] = [\partial\alpha_A]$ as the homology class corresponding to the boundary of the chain α_A.*

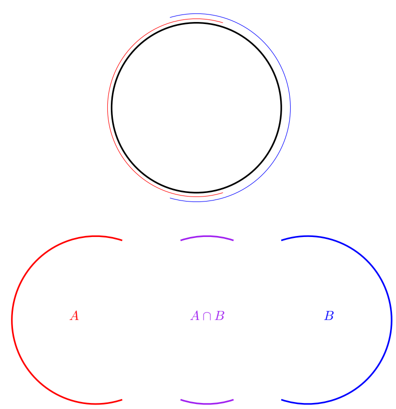

Figure 9: A decomposition of S^1 into two intervals.

Example 2.4. *We will compute the homology of S^1 with coefficients in a field $\mathbb{F}$. Express S^1 as the union of two intervals A and B as Figure*

9 suggests. The only non-trivial part of the corresponding Mayer-Vietrois sequence is the following:

$$H_1(A) \oplus H_1(B) \to H_1(X) \to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(X) \to 0,$$

which is of the form

$$0 \to H_1(X) \xrightarrow{\delta_1} \mathbb{F}^2 \xrightarrow{(i_0,j_0)} \mathbb{F}^2 \xrightarrow{\mu_0} H_0(X) \xrightarrow{\delta_0} 0.$$

We proceed by the following sequence of deductions:

1. $\operatorname{rank} \delta_0 = 0$ *as it is the trivial map.*
2. *Map* μ_0 *is of rank*[17] 1.
3. *By Proposition 2.2 we get* $\dim H_0(S^1) = 1$.
4. *By exactness and observation 2. we have* $\dim \operatorname{Im}(i_0, j_0) = \dim \ker \mu_0 = 1$, *hence* $\operatorname{rank}(i_0, j_0) = 1$.
5. *By exactness and the previous item we have* $\dim \operatorname{Im} \delta_1 = \dim \ker(i_0, j_0) = 1$, *hence* $\operatorname{rank} \delta_1 = 1$.
6. *By Proposition 2.2 we get* $\dim H_1(S^1) = 1$.
7. *All higher homotopy groups (for* $n > 1$*) are trivial as they appear as* $\cdots 0 \to H_n(X) \to 0 \cdots$ *in the exact sequence which, by Proposition 2.2, means they are trivial.*

[17] Recall that $\mu(u, v) = u - v$. Its rank is either $0, 1$, or 2. Its can't be 0 as the map is nontrivial. It can't be 2, as it has a non-trivial kernel generated by (u, u) since A and B are in the same component of X.

Remark 2.5. *In the same manner we could compute the homology groups of* S^m *for each* m *by observing that it can be decomposed as the union of two hemispheres (*m*-discs) whose intersection is homotopy equivalent to* S^{m-1}*, see Figure 10.*

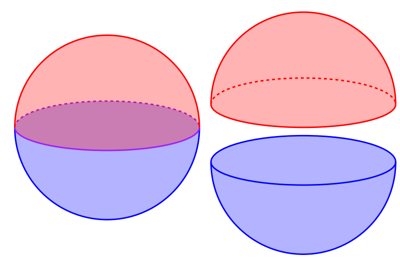

Figure 10: A decomposition of S^2 into two discs, whose intersection is S^1.

Example 2.6. *In a similar manner we can compute the homology of a two-dimensional torus* X *with coefficients in any field* $\mathbb{F}$*. We will only mention how to compute its first homology as the homology groups of other dimensions are already familiar to us.*

We will use the decomposition of Figure 11. The relevant part of the Mayer-Vietoris sequence is

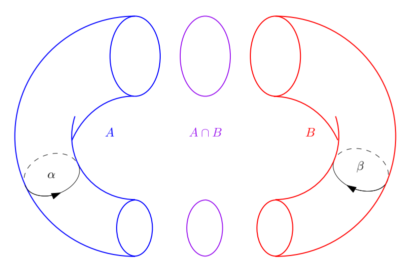

Figure 11: A decomposition of the torus into two parts, whose intersection is the disjoint union of two copies of S^1.

$$H_1(A) \oplus H_1(B) \to H_1(X) \to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(X) \to 0$$

which is of the form

$$\mathbb{F}^2 \xrightarrow{\mu_1} H_1(X) \xrightarrow{\delta_1} \mathbb{F}^2 \xrightarrow{(i_0,j_0)} \mathbb{F}^2 \to \mathbb{F} \to 0.$$

We proceed by the following sequence of deductions:

1. *By the same argument as in Example 2.4 we have* $\operatorname{rank} \delta_1 = 1$.

2. *The generators of* $H_1(A)$ *and* $H_1(B)$ *are cycles/loops* α *and* β *respectively. Note that* $\alpha \simeq \beta$ *in* X *thus* $[\alpha] = [\beta] \in H_1(X)$*. Furthermore, as*[18] $0 \neq [\alpha] \in H_1(X)$*, we have*[19] $\operatorname{rank} \mu_1 = 1$.

3. *By Proposition 2.2 we get* $\dim H_1(X) = 2$.

3 Concluding remarks

Recap (highlights) of this chapter

- Induced maps on homology and functoriality;
- Brouwer fixed point Theorem and Hairy ball Theorem;
- Exact sequences;
- Mayer-Vietoris exact sequence.

Background and applications

Invariants of homological nature appear throughout topology, geometry and other fields of mathematics. The examples of theoretical applications presented here barely scratch the surface. Some of the settings in which such constructions contributed to significant development include knot theory (Khovanov homology), differential geometry (De Rham cohomology, Floer homology), etc.

The Mayer-Vietoris sequence arises from a decomposition of a space into two pieces. A natural question about a similar result in the context of decompositions into more pieces in treated within the context of spectral sequences, an algebraic formalism far above the reach of our presentation. These theoretical developments allow for a certain level of distributed computation of homology.

Appendix: Zig-Zag Lemma

The Mayer-Vietoris long exact sequence is derived from the Zig-Zag Lemma.

Lemma 3.1. *[Zig-Zag Lemma] Let* $\mathbb{F}$ *be a field of coefficients. Assume the following diagram of vector spaces over* $\mathbb{F}$ *and linear maps*[20] *is commutative*[21]*:*

[18] An algebraic way to see that $0 \neq [\alpha] \in H_1(X)$ is through the Mayer-Vietrois sequence:

Proof. If $[\alpha]$ was trivial in $H_1(X)$ then $([\alpha], 0)$ would have been in $\ker \mu_1$. By exactness, this would mean that $([\alpha], 0) \in \operatorname{Im}(i_1, j_1)$. However, $\operatorname{Im}(i_1, j_1)$ is generated by the images of the two obvious cycles in $A \cap B$, each of which maps into $(\pm[\alpha], \mp[\beta])$. Space $\operatorname{Im}(i_1, j_1)$ is thus one-dimensional and generated by $(\pm[\alpha], \mp[\beta])$, hence $([\alpha], 0) \notin \operatorname{Im}(i_1, j_1)$ as $[\alpha] \neq 0$ in $H_1(A)$ and $[\beta] \neq 0$ in $H_1(B)$. $\square$

[19] The map μ_1 is defined as $\mu_1(u, v) = u - v$ in the basis $([\alpha], 0), (0, [\beta])$ of $H_1(A) \oplus H_1(B)$. Its rank is either $0, 1$, or 2. Its can't be 0 as the map is nontrivial, since α is not nullhomologous in X. It can't be 2, as it has a non-trivial kernel generated by (u, u) since $[\alpha] = [\beta] \in H_1(X)$.

[20] For the sake of simplicity the indices of maps will be omitted. For example, maps $\alpha_q \colon A_q \to B_q$ are all denoted by α even though they depend on q . For the same reason we will refrain from mentioning $\mathbb{F}$ again.

[21] I.e., $\partial \circ \alpha = \alpha \circ \partial$ and $\partial \circ \beta = \beta \circ \partial$.

$$
\begin{tikzcd}
 & \vdots \arrow[d, "\partial"] & \vdots \arrow[d, "\partial"] & \vdots \arrow[d, "\partial"] & \\
0 \arrow[r] & A_{q+1} \arrow[r, "\alpha"] \arrow[d, "\partial"] & B_{q+1} \arrow[r, "\beta"] \arrow[d, "\partial"] & C_{q+1} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_q \arrow[r, "\alpha"] \arrow[d, "\partial"] & B_q \arrow[r, "\beta"] \arrow[d, "\partial"] & C_q \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_{q-1} \arrow[r, "\alpha"] \arrow[d, "\partial"] & B_{q-1} \arrow[r, "\beta"] \arrow[d, "\partial"] & C_{q-1} \arrow[r] \arrow[d, "\partial"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

☞ An exact sequence of the form

$$0 \to A \to B \to C \to 0$$

is called a **short exact sequence**. In such a situation, map $A \to B$ is injective as its kernel is the trivial image of the map $0 \to A$. On a similar note, $B \to C$ is surjective as its image is the kernel of the map $C \to 0$, which is C. As $\mathrm{Im}(B \to C) \cong B/\ker(B \to C)$ we conclude $C \cong B/A$ since equality $\ker(B \to C) = \mathrm{Im}(A \to B)$ holds by exactness.

If each row is a short exact sequence, and each columns is a chain complex[22]*, then there exists a long exact sequence of homology groups*[23]

[22] I.e., $\partial^2 = 0$.

[23] I.e., the homology groups arising from the vertical chain complexes. In particular, $H_q(A)$ is the quotient

$$\ker(A_q \to A_{q-1}) / \mathrm{Im}(A_{q+1} \to A_q).$$

$$
\begin{tikzcd}
\cdots \arrow[r, "\alpha_*"] & H_{q+1}(B) \arrow[r, "\beta*"] & H_{q+1}(C) \arrow[dll, "\delta"'] \\
H_q(A) \arrow[r, "\alpha_*"] & H_q(B) \arrow[r, "\beta*"] & H_q(C) \arrow[dll, "\delta"'] \\
H_{q-1}(A) \arrow[r, "\alpha_*"] & H_{q-1}(B) \arrow[r, "\beta*"] & \cdots
\end{tikzcd}
$$

The idea of a proof. The proof is performed using the “diagram chasing” technique. We will only prove the existence of the δ map.

In order to define δ let us choose a non-trivial cycle $c \in C_{q+1}$. Charted by the diagrams on the right, the chase after $\delta([c])$ begins:

Diagram 1: $\partial(c) = 0$ as c is a cycle.

Diagram 2: By the exactness of the row map β is surjective, thus there exists $b_1 \in \beta^{-1}(c)$. Define $b_2 = \partial(b_1)$. By the commutativity $\beta(b_2) = 0$.

Diagram 3: By the exactness of the row map there exists $a_1 \in \alpha^{-1}(b_2)$. **Define** $\delta([c]) = [a_1]$.

The rest of the proof goes along the same lines. For example, in order to prove a_1 is a cycle we use diagram 4:

- Define $a_2 = \partial(a_1)$ observe $\partial(b_2) = 0$ as $\partial^2 = 0$.
- By the commutativity $\alpha(a_2) = 0$.
- By the exactness of the row $a_2 = 0$, hence a_1 is a cycle. $\square$

☞ Diagram 1:

$$
\begin{tikzcd}
c \arrow[r, mapsto] \arrow[d, mapsto, "\partial"] & 0 \\
0 &
\end{tikzcd}
$$

☞ Diagram 2:

$$
\begin{tikzcd}
b_1 \arrow[r, mapsto, "\beta"'] \arrow[d, mapsto, "\partial"] & c \arrow[r, mapsto] \arrow[d, mapsto] & 0 \\
b_2 \arrow[r, mapsto, "\beta"] & 0 &
\end{tikzcd}
$$

☞ Diagram 3:

$$
\begin{tikzcd}
 & b_1 \arrow[r, mapsto] \arrow[d, mapsto] & c \arrow[r, mapsto] \arrow[d, mapsto] & 0 \\
a_1 \arrow[r, mapsto, "\alpha"] & b_2 \arrow[r, mapsto] & 0 &
\end{tikzcd}
$$

☞ Diagram 4:

$$
\begin{tikzcd}
 & b_1 \arrow[r, mapsto] \arrow[d, mapsto] & c \arrow[r, mapsto] \arrow[d, mapsto] & 0 \\
a_1 \arrow[r, mapsto] \arrow[d, mapsto] & b_2 \arrow[r, mapsto] \arrow[d, mapsto] & 0 & \\
a_2 \arrow[r, mapsto] & 0 & &
\end{tikzcd}
$$

Remark 3.2. *The construction and proof of the Mayer-Vietoris sequence follows from the Zig-Zag Lemma using the following*[24] *commutative diagram*

[24] We will be using the notation of Theorem 2.3.

$$\begin{tikzcd}
 & \vdots \arrow[d, "\partial"] & \vdots \arrow[d, "\partial"] & \vdots \arrow[d, "\partial"] & \\
0 \arrow[r] & C_{q+1}(A \cap B) \arrow[r, "\alpha"] \arrow[d, "\partial"] & C_{q+1}(A) \oplus H_{q+1}(B) \arrow[r, "\beta"] \arrow[d, "\partial"] & C_{q+1}(X) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_q(A \cap B) \arrow[r, "\alpha"] \arrow[d, "\partial"] & C_q(A) \oplus H_q(B) \arrow[r, "\beta"] \arrow[d, "\partial"] & C_q(X) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_{q-1}(A \cap B) \arrow[r, "\alpha"] \arrow[d, "\partial"] & C_{q-1}(A) \oplus H_{q-1}(B) \arrow[r, "\beta"] \arrow[d, "\partial"] & C_{q-1}(X) \arrow[r] \arrow[d, "\partial"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

with maps α being induced by inclusion, and maps β being the subtraction maps[25]*. Observe that the horizontal maps are short exact sequences.*

[25] I.e., $\beta([\gamma_1], [\gamma_2]) = [\gamma_1 - \gamma_2]$.

Zig-Zag Lemma provides a useful template for constructions of exact sequences. Another setting in which it applies is that of relative homology.

Appendix: Relative homology

Let us fix a field $\mathbb{F}$, a simplicial complex K, and $L \leq K$. Homology construction on K is based on cycles: chains whose boundaries are trivial. The concept of relative homology expands this construction in the following way: given $L \leq K$, the relative homology construction is based on relative cycles, i.e., chains, whose boundaries are contained in L.

Algebraic specifics of the definition. From the chain complexes of K and L we can construct the quotient chain complex:

$$\cdots \xrightarrow{\partial} C_q(K)/C_q(L) \xrightarrow{\partial} C_{q-1}(K)/C_{q-1}(L) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(K)/C_0(L) \xrightarrow{\partial} 0$$

The **relative homology groups** $H_q(K, L)$ are the homology groups arising from this chain complex. In particular[26]:

[26] In this occurrence we also mention the coefficients for the sake of a complete defining formula.

$$H_q(K, L; \mathbb{F}) = \frac{\ker\left(C_{q+1}(K)/C_{q+1}(L) \xrightarrow{\partial} C_q(K)/C_q(L)\right)}{\operatorname{Im}\left(C_q(K)/C_q(L) \xrightarrow{\partial} C_{q-1}(K)/C_{q-1}(L)\right)}.$$

☞ Observe that $H_q(K, \emptyset) = H_q(K)$.

Combining Lemma 3.1 and the commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d, "\partial"] & \vdots \arrow[d, "\partial"] & \vdots \arrow[d, "\partial"] & \\
0 \arrow[r] & C_{q+1}(L) \arrow[r, hook] \arrow[d, "\partial"] & C_{q+1}(K) \arrow[r] \arrow[d, "\partial"] & C_{q+1}(K)/C_{q+1}(L) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_q(L) \arrow[r, hook] \arrow[d, "\partial"] & C_q(K) \arrow[r] \arrow[d, "\partial"] & C_q(K)/C_q(L) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_{q-1}(L) \arrow[r, hook] \arrow[d, "\partial"] & C_{q-1}(K) \arrow[r] \arrow[d, "\partial"] & C_{q-1}(K)/C_{q-1}(L) \arrow[r] \arrow[d, "\partial"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

we conclude that relative homology groups fit into the following exact sequence:

$$\cdots \to H_{q+1}(K, L) \to H_q(L) \to H_q(K) \to H_q(K, L) \to \cdots$$
$$\cdots \to H_0(L) \to H_0(K) \to H_0(K, L) \to 0,$$

Relative homology has a geometric meaning, which expands that of the usual homology. Table 1 summarizes the relative homology of the pair (K, L) of simplicial complexes from Figure 12.

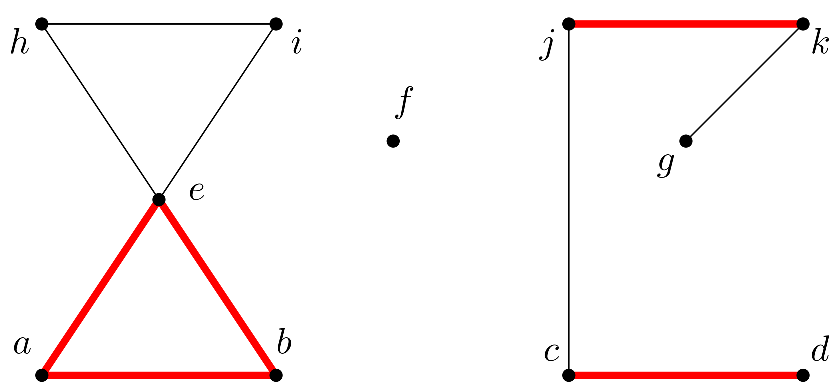

Figure 12: Simplicial complex K. Its subcomplex $L \leq K$ contains vertices a, c, b, d, e, j, k and all edges between these vertices. It is depicted by bold red edges.

q	$\dim H_q(K)$	$\dim H_q(K, L)$
0	3	1
1	2	2

Table 1: The comparison of the homology of K and the relative homology of the pair (K, L) from Figure 12 .

Let us geometrically interpret Table 1:

Dimension 0*:* K has three components. However, the relative homology detects only component $[f]$. Homology class $[e]$ is contained in L and is thus trivial by the definition. Homology class $[h]$ is homologous to $[e]$ and thus trivial as well.

Dimension 1*:* A natural basis for $H_1(K)$ would consist of $[\langle a,b\rangle + \langle b,e\rangle + \langle e,a\rangle]$ and $[\langle e,i\rangle + \langle i,h\rangle + \langle h,e\rangle]$. A natural basis for $H_1(K,L)$ however would consist of $[\langle e,i\rangle + \langle i,h\rangle + \langle h,e\rangle]$ and $[\langle i,j\rangle]$. Note that:

- $[\langle a,b\rangle + \langle b,e\rangle + \langle e,a\rangle]$ is a trivial in $H_1(K,L)$ as it is contained in L.
- $\langle i,j\rangle$ is a cycle in the relative homology chain complex as its boundary is contained in L and thus trivial.

Geometrically we can think of the relative homology $H_*(K,L)$ as the homology of the space obtained from K when the subcomplex L is contracted to a point, see Figure 13 for an example. The only exception to this rule is $H_0(K,L)$, whose dimension is one less[27] than the number of the components of the resulting space[28].

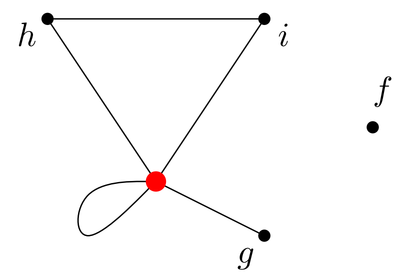

Figure 13: The space obtained from simplicial complex K from Figure 12 by contracting the subcomplex L to a point. The space has two holes but is not a simplicial complex in general.

[27] In the literature this exception is usually encoded in the phrase "reduced homology".

[28] Note that the resulting space does not inherit the structure of a simplicial complex from K. However, it can be triangulated.