# *Surfaces*

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Surfaces are some of the simplest topological spaces appearing frequently in science and data analysis. From each local perspective appearing as a part of the plane, the global shape of a surface may take many forms. Think of the surface of the earth: because it appears to be “planar” at each point, it had long been believed that Earth is actually a part of a plane or maybe a disc instead of a sphere.

Surprisingly enough, most surfaces of interest can be topologically recognized fairly easily. In this lecture we explain this recognition process and the accompanying theory, both of which will come handy in the lectures to come.

## 1 Surfaces as manifolds

Ever since the ancient times people were wondering about the shape of the world. They agreed that from the perspective of a human being, the world looked like a plane, a part of a surface. What was much harder to figure out was the global picture. The first and most obvious idea was that the world was a flat disc. Later came indications, such as deviations in the angle of the shadow depending on the latitude, that the world might be curved. Magellan’s first circumnavigation of the Earth does not constitute a rigorous proof that the world is a sphere by modern mathematical standards, but at the time it was a momentous achievement which confirmed that the Earth is indeed round.

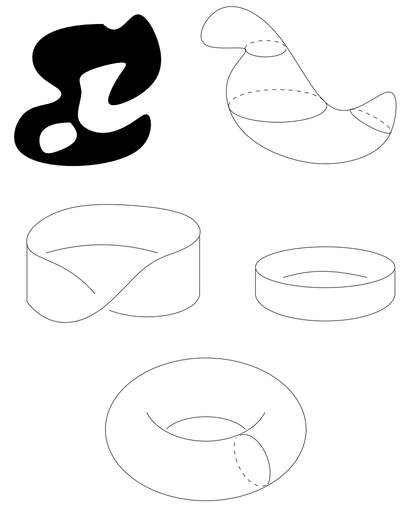

Figure 1: Some of the surfaces we mentioned before: a planar set (top left), a space homeomorphic to $S^2$ (top right), Moebius band (center left) and the usual band (center right), torus (bottom).

While we will not be sailing around the world in this course, we will be interested in the moral of this story: things that locally look like a plane may globally not be a plane. We will want to determine the global structure from local information.

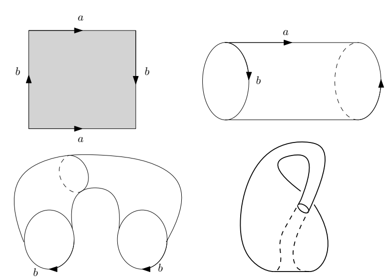


Figure 2: Klein bottle obtained by identifying the edges of a square: two along the same direction and two along the opposite direction. The resulting space (bottom right) is not realizable in $\mathbb{R}^3$ due to the self-intersection. However, Klein bottle can be embedded into $\mathbb{R}^4$ without self-intersections.

Spaces that locally look like a plane are called surfaces and their generalizations to other dimensions are called manifolds. Here is a formal definition.

**Definition 1.1.** *Let $n \in \{0, 1, 2, \dots\}$. A metric space $X$ is an $n$-**manifold**, if for each $x \in X$ there exists $r > 0$, such that $B_X(x, r)$ is homeomorphic to the $n$-dimensional disc $D^n$.*

2-manifolds are called **surfaces**.

Point $x$ on an $n$-manifold $X$ is:

- a **boundary point** if a homeomorphism $B_X(x, r) \to D^n$ from Definition 1.1 maps $x$ to a point on the boundary of $D^n$.

- an **interior point** if a homeomorphism $B_X(x, r) \to D^n$ from Definition 1.1 maps $x$ to a point in the interior of $D^n$.

These two notions are independent of the choice of homeomorphism $B_X(x, r) \to D^n$. Each point of $X$ is either a boundary point or an interior point. The **boundary** of $X$ consists of all the boundary points, and the **interior** of $X$ consists of all the interior points.

We say that a manifold $X$ is **without boundary**, if it has no boundary points. For an $n$-manifold $Y$ its boundary is an $(n-1)$-manifold without boundary, as can be seen from the examples below. For our purposes a **closed manifold** (closed surface) will be a manifold (surface) without boundary admitting a (finite) triangulation.

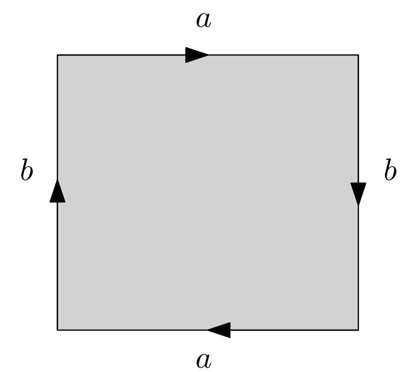


Figure 3: Projective plane obtained by identifying the edges of a square: both pairs along the opposite direction. The resulting space is not realizable in $\mathbb{R}^3$. However, it can be embedded into $\mathbb{R}^4$ without self-intersections.

**Example 1.2.** *We provide some examples of connected $n$-manifolds listed by dimension $n$.*

- $n = 0$*: This one is fairly unimpressive: a single point.*
- $n = 1$*: Circle and intervals $(0, 1), [0, 1], (0, 1]$. Each connected 1-manifold is homeomorphic to one of these. A circle and an open interval have no boundary, while the boundary of $[0, 1]$ consists*[1] *of 0 and 1.*
- $n = 2$*: We will provide a list of all surfaces by the end of this lecture. Here we list some of the more prominent ones. The already mentioned ones are recapped in Figure 1: note that the boundary of the band consists of two copies of $S^1$, while the Moebius band has a single boundary component. A closed disc $D^2$ is also a surface, whose boundary is $S^1$.*

  *Closely related to the torus are the* ***Klein bottle*** *(see Figure 2) and the* ***projective plane****, neither of them has a boundary and neither can be obtained as a subset of $\mathbb{R}^3$. However, they can be obtained as subsets of $\mathbb{R}^4$. While these two spaces are challenging to imagine geometrically, it is fairy easy to provide their (abstract) triangulations (see Figure 2) and compute some of their topological invariants, such as the Euler characteristic. The Torus and the Klein bottle have Euler characteristic 0, while the Euler characteristic of the projective plane is 1.*

  *The projective space is homeomorphic to the space of all 1-dimensional subspaces in $\mathbb{R}^3$.*

- *General $n$: $D^n$ and $S^n$ are both $n$-manifolds. The boundary of $D^n$ is $S^{n-1}$, while $S^n$ has no boundary.*

[1] Also, the boundary of $[0, 1)$ is 0.

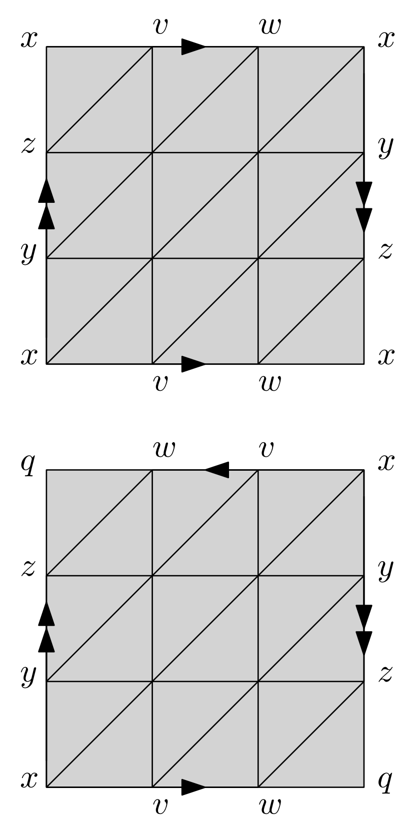


Figure 4: Triangulations of the Klein bottle (top) and the projective plane (bottom).

## *Combinatorial manifolds*

We will mostly be working with triangulated manifolds. A natural question that arises in this context is how to recognize whether a given simplicial complex is a triangulation of a manifold. Tackling this task we first introduce nice combinatorial descriptions of manifolds.

> **Definition 1.3.** *Suppose $K$ is a simplicial complex and $n \in \mathbb{N}$. We say that $K$ is a* ***combinatorial*** $n$***-manifold****, if for each vertex $v \in K$ its link* $\mathrm{Lk}(v)$ *is homeomorphic either to $S^{n-1}$ or $D^{n-1}$.*

Properties and notation:

- Each combinatorial $n$-manifold is a triangulation of an $n$-manifold.
- For $n < 4$, each $n$-manifold admits a triangulation as a combinatorial $n$-manifold[2].
- Vertices of a combinatorial manifold $K$ satisfying $\mathrm{Lk}(v) \cong B^{n-1}$ are called **boundary vertices**.
- Vertices of a combinatorial manifold $K$ satisfying $\mathrm{Lk}(v) \cong S^{n-1}$ are called **interior vertices**.
- Edges of a combinatorial surface $K$ that are contained in only one triangle are called **boundary edges**. The union of the boundary edges corresponds to the boundary of the manifold.
- Edges of a combinatorial surface $K$ that are contained in two triangles are called **interior edges**. No edge in a combinatorial surface is contained in more than two triangles.

[2] Surprisingly enough, this does not hold for $n \geq 4$

Using these properties it is fairly easy to recognize whether a given simplicial complex $K$ is a combinatorial surface and thus a triangulation of a surface: for each vertex $v \in K$ we verify whether $\mathrm{Lk}(v)$ is homeomorphic to $S^1$ or $D^1$. It is easy to see that a connected 1-dimensional simplicial complex is homeomorphic to:

- $S^1$ iff each of its vertices is contained in two edges.
- $D^1$ iff two of its vertices are contained in one edge, and all other vertices are contained in two edges.

We will leave the elementary proofs of these two facts to the reader.

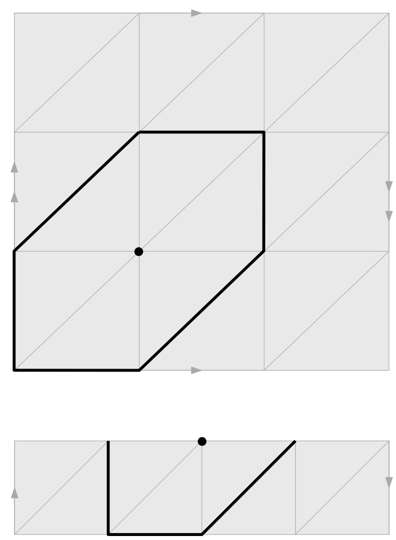

Figure 5: Triangulation of the Klein bottle (top) and of the Moebius band (bottom). In the top triangulation each vertex is an interior vertex as each link (bold) is homeomorphic to $S^1$. In the bottom case each vertex is a boundary vertex as each link (bold) is homeomorphic to $B^1$.

## 2 Orientability

Orientability is about defining “up and down”. It is quite easy to agree on the two directions on the surface of the earth. However, in general that may not be the case for all surfaces. Consider a usual band from Figure 1. It is orientable because we can define two different sides of it. To put it into a more colorful language, we can color one side of the band in red and the other side in blue, without the colors ever touching each other. The story is different on the Moebius band : it has only one side. We could start coloring it in red at some spot and keep expanding the color along the surface (but not across the boundary): eventually we will color the whole band, i.e., there is no “other” side.

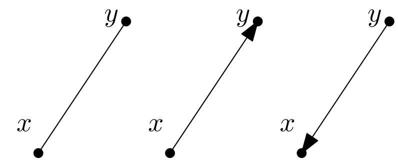


Figure 6: The edge $\{x,y\}$ (left) and the oriented edges $\langle x,y\rangle$ (center) and $\langle y,x\rangle = -\langle x,y\rangle$ (right).

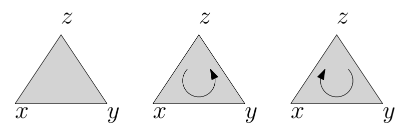


Figure 7: The triangle $\{x,y,z\}$ (left) and the oriented triangles $\langle x,y,z\rangle = \langle y,z,x\rangle = \langle z,x,y\rangle$ (center) and $\langle y,x,z\rangle = \langle x,z,y\rangle = \langle z,y,x\rangle = -\langle x,y,z\rangle$ (right).

Orientability is an important property of surfaces. It will be required for our classification result. In order to fully understand it we have to define orientation for simplices first. Besides its application in this section orientation on simplices will feature prominently later within the context of homology computation.

Up to now a simplex was given by a set of its vertices. An oriented simplex is a simplex with a choice of orientation. For an edge that means direction, for a triangle that means “a normal” (see Figures 6 and 7). This direction/orientation will be described by a choice of an order on vertices.

**Definition 2.1.** *An* ***oriented simplex*** *on vertices* $v_0, v_1, \ldots, v_k$ *is an ordered* $(k+1)$*-tuple* $\sigma = \langle v_0, v_1, \ldots, v_k\rangle$. *For a permutation* $\pi$ *on* $\{0,1,\ldots,k\}$ *we identify:*

$$\sigma = (-1)^{\mathrm{sgn}(\pi)} \langle v_{\pi(0)}, v_{\pi(1)}, \ldots, v_{\pi(k)}\rangle,$$

*where* $\mathrm{sgn}(\pi)$ *is the signature of permutation* $\pi$, *i.e., value* $0$ *if* $\pi$ *is even and value* $1$ *if* $\pi$ *is odd.*

*A* $0$*-dimensional simplex with vertex* $v$ *can also be oriented in two ways: as* $\langle v\rangle$ *and as* $-\langle v\rangle$.

Figures 6 and 7 provide examples of descriptions of oriented edges and triangles, and their geometric interpretations. Here are some properties that follow from Definition 2.1:

- Each simplex on vertices $v_0, v_1, \ldots, v_k$ can be oriented in two different ways: $\sigma = \langle v_0, v_1, \ldots, v_k\rangle$ and $-\sigma$.
- An oriented simplex has a sign $+$ or $-$ prepended.
- Exchanging two vertices in an oriented simplex $\tau$ changes the orientation of $\tau$ by changing the prefixed sign.

An important property of an oriented simplex is that it induces an orientation on each of its facets.

**Definition 2.2.** *Suppose* $\sigma = \langle v_0, v_1, \ldots, v_k\rangle$ *is an oriented simplex and* $p \in \{0,1,\ldots,k\}$. *Then the* ***induced orientation*** *on the facet*

*$\sigma$ of obtained by dropping $v_p$ is*

$$(-1)^p \langle v_0, v_1, \ldots, v_{p-1}, v_{p+1}, \ldots, v_k \rangle.$$

Oriented edge $\langle x, y\rangle$ induces orientations $\langle y\rangle$ and $-\langle x\rangle$ on its facets (vertices). Oriented triangle $\langle x, y, z\rangle$ induces orientations $\langle y, z\rangle, -\langle x, z\rangle$ and $\langle x, y\rangle$ on its facets (edges), see Figure 8.

Now that we established a way to orient a single simplex, we turn our attention to orienting the whole surface.

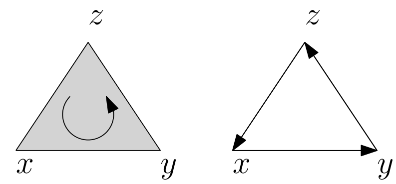


Figure 8: Oriented triangle $\langle x, y, z\rangle$ (left) and induced orientation on the edges (right): $\langle x, y\rangle, \langle y, z\rangle$, and $\langle z, x\rangle$. Note that the edges are oriented along the direction of the circular arrow indicating the orientation of the triangle.

**Definition 2.3.** *Suppose oriented* 2*-simplices $\sigma$ and $\sigma'$ share a common edge. Simplices $\sigma$ and $\sigma'$ are* ***oriented consistently****, if they induce the opposite orientation on the common edge. (see Figure 9)*

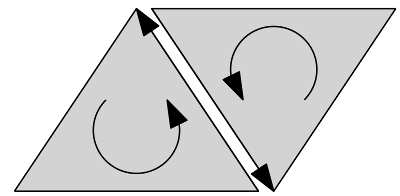

Figure 9: Consistent orientation: note that the orientations of the triangles agree (both directed circular arrows point counter-clockwise). This implies that the induced orientations on the common edge are opposite to each other.

**Definition 2.4.** *Let K be a triangulation of a surface $|K|$. We say that $|K|$ is* ***oriented****, if all triangles of K are oriented (as simplices) so that the following holds: each pair of oriented triangles with a common edge is oriented consistently.*

*A surface is* ***orientable*** *if it can be oriented.*

Orientability of a surface does not depend on a triangulation but on the topological type of the surface only. The following are two basic examples that demonstrate the underlying geometric idea.

**Example 2.5.** *The usual band $S^1 \times [0, 1]$ is orientable as Figure 10 demonstrates. To the contrary, the Moebius band is not orientable as Figure 11 demonstrates. Since the Klein bottle and the projective plane both contain a copy of the Moebius band (any of the three horizontal strips of triangulations in Figure 4), neither of them is orientable.*

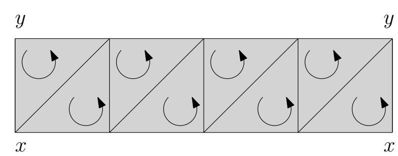


Figure 10: Orientable triangulation of a usual band. The oriented simplices induce the opposite orientation on all the edges, including the edge $\{x, y\}$, along which the glueing occurs.

As Example 2.5 and Figure 11 suggest it is fairly easy to check whether a connected triangulated surface is orientable. This can be done directly by orienting one triangle and then inductively orienting all neighboring triangles with shared edges, while checking that each newly oriented triangle is oriented consistently with respect to the already oriented triangles.

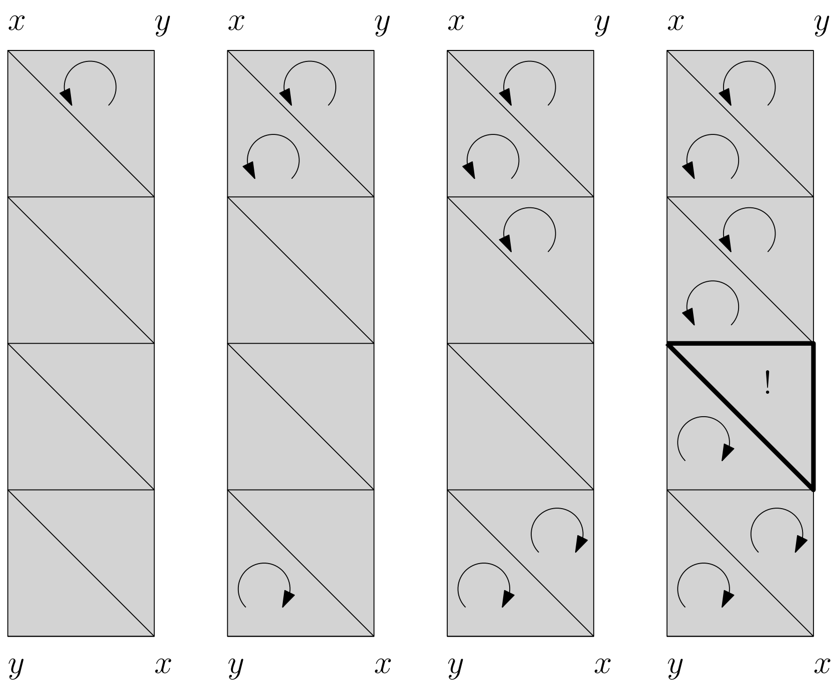


Figure 11: A proof that the Moebius band is not orientable. Assume we want to orient a triangulation of the Moebius band on the left. We first choose an orientation of one triangle (far left) and then inductively induce consistent orientation on the neighboring triangles. In the end (far right) we obtain conflicting requirements on the orientation on the last (bold) triangle, which means there is no consistent way to orient all the triangles in this triangulation.

## 3 Connected sum of surfaces

One of the basic operations on surfaces (and actually on manifolds in general) is the connected sum.

> **Definition 3.1.** *Suppose $X$ and $Y$ are connected surfaces. Choose topological 2-discs $D_X \subset X$ and $D_Y \subset Y$, neither of which contains any boundary point of the surfaces. The corresponding boundaries of these discs are topological 1-spheres (circles) $S_X \subset X$ and $S_Y \subset Y$ respectively. The* ***connected sum*** *$X \# Y$ is obtained by removing the interiors of discs $D_X$ and $D_Y$ from $X$ and $Y$, and gluing the resulting spaces by identifying $S_X$ with $S_Y$.*

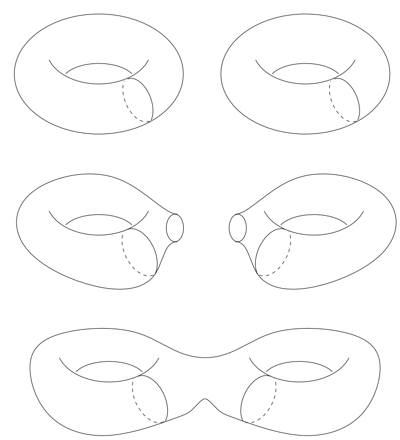

Figure 12: Two tori (top) and their connected sum (bottom) obtained by identifying the boundaries of the two removed discs (center).

See Figure 12 for a sketch of this construction. A few technical remarks about connected sums as defined above:

- It turns out that the topological type of $X \# Y$ does not depend on the choice of discs $D_X, D_Y$.
- A connected sum is a surface, whose boundary components correspond to the union of the boundary components of $X$ and the boundary components of $Y$.
- Surfaces $X$ and $Y$ are both orientable iff $X \# Y$ is orientable.

- For each surface $X$, the following holds: $X\#S^2 \cong X$.

If abstract simplicial complexes $K$ and $L$ are triangulations of surfaces $X$ and $Y$ respectively and $K \cap L = \emptyset$, we can obtain a triangulation $M$ of $X\#Y$ in the following way:

1. Choose triangles $\Delta_X$ and $\Delta_Y$ in $K$ and $L$ respectively, so that no point of these two triangles lies on the boundary of $X$ or $Y$.

2. Define
$$M = (K \setminus \{\Delta_X\}) \cup (L \setminus \{\Delta_Y\}) / \sim,$$
where $\sim$ stands for the identification of each of the boundary edges of $\Delta_X$ with an appropriate boundary edge of $\Delta_Y$.

In short, $M$ is obtained by removing $\Delta_X$ and $\Delta_Y$ from the union of $K$ and $L$, and then identifying the boundaries of the removed triangles. This procedure is a discrete version of the one in Definition 3.1.

**Proposition 3.2.** $\chi(X\#Y) = \chi(X) + \chi(Y) - 2$.

*Proof.* Assume abstract simplicial complexes $K$ and $L$ are triangulations of surfaces $X$ and $Y$ respectively and $K \cap L = \emptyset$. It is obvious that $\chi(K \cup L) = \chi(K) + \chi(L)$. In order to obtain a triangulation of $X\#Y$ from $K \cup L$, we:

- Remove two triangles (change $-2$ to the Euler characteristic);
- Identify three pairs of vertices, meaning we have three vertices less (change $-3$ to the Euler characteristic);
- Identify three pairs of edges, meaning we have three edges less (change $+3$ to the Euler characteristic);

The total change to the Euler characteristic after these steps is $-2$. □

## 4 Classification of surfaces

We can now describe the classification of surfaces. Let $\mathbb{T}$ denote the torus and let $\mathbb{P}$ denote the projective plane.

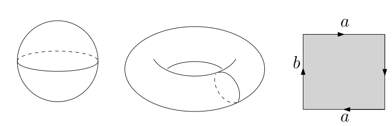


Figure 13: The three closed connected surfaces (the sphere $S^2$, the torus $\mathbb{T}$ and the projective plane $\mathbb{P}$), that are used to construct any other closed connected surface using the connected sum operation.

**Theorem 4.1.** *[Classification Theorem for closed connected surfaces] Suppose X is a closed connected surface. Then X is homeomorphic to one of the following:*

*1.* $S^2$.

*2.* *n-torus* $n\mathbb{T} = \underbrace{\mathbb{T}\#\mathbb{T}\#\ldots\#\mathbb{T}}_{n}$ *for some* $n \in \mathbb{N}$.

*3.* $n\mathbb{P} = \underbrace{\mathbb{P}\#\mathbb{P}\#\ldots\#\mathbb{P}}_{n}$ *for some* $n \in \mathbb{N}$.

It turns out that the surfaces appearing in Theorem 4.1 can be distinguished using orientability and the Euler characteristic. From the properties of the connected sum recall that (for each $n \in \mathbb{N}$) $S^2$ and $n\mathbb{T}$ are orientable while $n\mathbb{P}$ are not. Furthermore, using Proposition 3.2 and we can inductively deduce[3] :

- $\chi(n\mathbb{T}) = 2 - 2n$ as $\chi(\mathbb{T}) = 0$.
- $\chi(n\mathbb{P}) = 2 - n$ as $\chi(\mathbb{P}) = 1$.

Consequently we obtain the following table.

| Surface | $\chi$ | |
|---|---|---|
| $S^2$ | 2 | orientable |
| $T$ | 0 | orientable |
| $nT$ | $2-2n$ | orientable |
| $P$ | 1 | not orientable |
| $nP$ | $2-n$ | not orientable |

Table 1: A list of closed connected surfaces along with their Euler characteristic and orientability.

Theorem 4.1 motivates the following **classification algorithm** for a closed connected surface given as an abstract simplicial complex[4] $K$:

1. Check for orientability of $K$.
2. Compute the Euler characteristic.
3. Consult Table 1.

**Example 4.2.** *Which of the surfaces in Theorem 4.1 is the Klein bottle? We have already discovered that it is not orientable and that its Euler characteristic is* 0*. By the Classification Theorem the Klein bottle is homeomorphic to* $\mathbb{P}\#\mathbb{P}$.

☞ A shortcut to computing $\chi$: while the Euler characteristic is formally defined on a triangulation, it turns out is can also be obtained from the representation of a surface in terms of a polygon with identified sides. For example, the representations of torus in Figure 13 and Klein bottle of Figure 2 have one 2-dimensional square, two 1-dimensional edges, and one vertex, yielding $\chi = 2$. The representation of the projective plane in Figure 3 has one 2-dimensional square, two 1-dimensional edges, and two vertices, yielding $\chi = 1$. This trick could assist with Figure 14. A justification will be provided in the context of discrete Morse theory.

### *General surfaces*

Theorem 4.1 can also be used to classify general surfaces admitting a finite triangulation. Suppose $X$ is a surface:

1. If $X$ is not connected, it is a disjoint union of connected surfaces and it obviously suffices to recognize each of its components.
2. If $X$ is connected and has a boundary $Y$, then $Y$ is a 1-manifold without boundary, meaning $Y$ is a disjoint union of $k$ copies of $S^1$ for some $k \in \mathbb{N}$. By glueing a disc along each component of $Y$ we obtain a closed connected surface $X'$, which we can recognize [5]. We conclude that $X$ is homeomorphic to $X'$ with $k$ discs removed[6].

[3] Using Proposition 3.2 we can deduce $\chi(2\mathbb{T}) = \chi(\mathbb{T}\#\mathbb{T}) = \chi(\mathbb{T}) + \chi(\mathbb{T}) - 2 = 0 + 0 - 2 = -2$, $\chi(3\mathbb{T}) = \chi(\mathbb{T}) + \chi(\mathbb{T}\#\mathbb{T}) - 2 = 0 + -2 - 2 = -4$, and proceed inductively. The proof for $n\mathbb{P}$ is analogous.

[4] I.e., we assume the triangulation $K$ is a connected combinatorial 2-manifold and has no boundary components.

[5] Such a gluing of discs does not change the orientation, i.e., $X$ is orientable iff $X'$ is. However, an addition of each disc increases the Euler characteristic by 1.

[6] It turns out that the homeomorphic type does not depend on the discs we remove from $X'$, only on their number.

**Example 4.3.** *It is easy to see that* $S^1 \times [0,1]$ *is obtained from* $S^2$ *by removing two discs. It is a bit harder to see how to get the Moebius band* $M$ *this way. It is easy to see that* $M$ *has one boundary component and has Euler characteristic*[7] $0$*. Gluing a disc along the boundary component we obtain a closed connected non-orientable surface of Euler characteristic* $1$*, which is* $\mathbb{P}$*. Hence the Moebius band is obtained by removing a disc from the projective plane.*

[7] We could count the simplices in Figure 11.

We are now ready to state a **classification algorithm** for a surface given as an abstract simplicial complex $K$:

1. Partition $K$ into its connected components and classify each of them.

2. For each component $K'$:

   (a) Count the number $n(K')$ of boundary components of $K'$.

   (b) Check for orientability of $K'$.

   (c) Compute the Euler characteristic of $K'$.

   (d) Let $Y$ be the surface matching the orientability of $K'$ and of Euler characteristic[8] $\chi(K') + n(K')$ by Table 1.

   (e) Surface $K'$ is homeomorphic to $Y$ with $n(K')$ many discs removed.

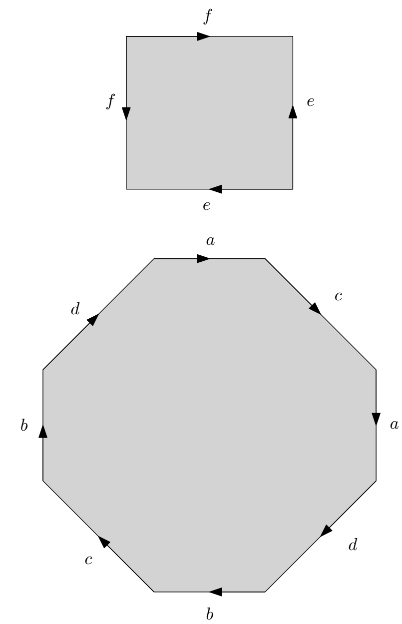


Figure 14: Which surfaces are these?

[8] $\chi(K') + n(K')$ is the Euler characteristic of a surface obtained from $K$ by gluing $n(K')$ discs along the boundary components of $K'$.

With this classification algorithm we can always determine whether two surfaces are homeomorphic or not.

## 5 Concluding remarks

### *Recap (highlights) of this chapter*

- Surfaces, combinatorial surfaces;
- Orientation and orientability;
- Connected sum of surfaces;
- Classification of surfaces;

### *Background and applications*

For most of the practical purposes, we live in a three-dimensional space. Objects in our everyday life are often modelled by surfaces enclosing the objects. Outputs of many 3-D scans are given in terms of triangulated surfaces (for example, as .stl files).

Surfaces and other higher-dimensional manifolds are also often assumed to be the underlying spaces in specific settings. A randomly generated bitmap image will seldom represent something reasonable, and yet there is a huge number of images that convey an information to the human eye. A space of "recognizable" images is a huge subspace (perhaps a manifold) in the space of all bitmap images. Manifold recognition approaches attempt to detect the underlying manifolds from sample data.

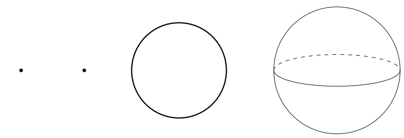

Figure 15: Spheres $S^0$ (two points), $S^1$ (a circle), and $S^2$ (a sphere).

The third source of surfaces and manifolds are spaces described with two or more degrees of freedom: configuration spaces of molecules, robotic arms, etc. For example, the configuration space of a robotic arm (i.e., the space of all possible positions of the arm) with two independent joins, each of which allows a full rotational motion, is the torus $S^1 \times S^1$. On a similar note, given two annotated[9] points on $S^1$, the configuration space of all possible positions of the two points is again a torus $S^1 \times S^1$, since the degree of freedom of each point is $S^1$. It is interesting to observe the difference that appears if the points are not annotated[10]: in such a case all possible configurations actually form the projective plane.

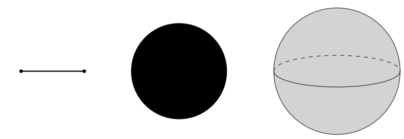

Figure 16: Discs $D^1$ (a line segment), $D^2$, and $D^3$.

[9] The annotation refers to the fact that each point has a name in the sense that if the two points lie at different positions, then exchanging them changes the configuration. The situation is sometimes also described using "ordered pairs" of points $(x, y)$ for which $x, y \in S^1$.

[10] In particular, if two non-annotated points lie at different positions, then exchanging them does not change the configuration as the pair represents the same collection of points on $S^1$. The situation is sometimes also described using "unordered pairs" of points $(x, y)$ by additionally identifying $(x, y) \sim (x', y')$ if $x = y'$ and $y = x'$.

On a more theoretical note, the question of whether each manifold admits a triangulation had been one of the focal points of topology in the previous century. It turns out that every manifold in dimension 3 or less admits a triangulation. Surprisingly enough, there are manifolds in higher dimensions that do not admit any triangulation.

## *Appendix: imagining* $S^3$

In this appendix we will try to explain two ways of thinking about the three-dimensional sphere $S^3$ and spheres in general.

1. The first observation has to do with the relationship between discs and spheres. We have already mentioned that $S^{n-1}$ appears as the boundary of $D^n$. It should also be apparent (see Figure 17) that gluing two copies of a disc $D^n$ along their boundaries (copies of $S^{n-1}$) results in $S^n$. In particular, we obtain $S^3$ by taking two 3-discs (solid balls) and gluing them along the boundary.

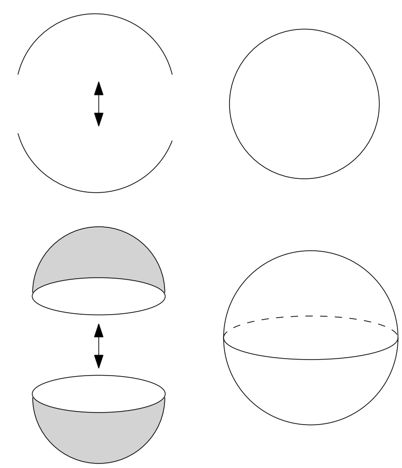

Figure 17: Gluing two copies of a disc together results in a sphere.

2. As for the second observation we will refer to Figure 18. It turns out that $S^n$ can be obtained in the following way: pick two opposite points (the north pole and the south pole) and span an interval's worth of copies of the sphere $S^{n-1}$ between them, so that the spheres are shrinking as they are approaching the poles.

   Note that if we only take one point and have the spheres shrinking only as they approach that point (and have them increase otherwise) we obtain $D^n$ (see Figure 19 for some low-dimensional examples).

It may come as a surprise that these points of view can be observed in Dante's *Divine Comedy*, written about seven centuries ago. Dante's description of the universe coincides with the topology of $S^3$: on one extreme are the depths of Hell (part of Inferno), from which Dante is guided by Beatrice through the spheres of Inferno, Purgatory, and Paradise, until he reaches Empyrean, the place which contains the essence of God.

This description coincides with 2. above in terms of spheres. Equivalently, we may consider Inferno and Purgatory together as one 3-disc with center at Hell, and Paradise as another 3-disc with center at Empyrean: in this setup the Universe consists of both 3-discs that intersect along the surface of the Earth (which coincides with the boundary $S^2$).

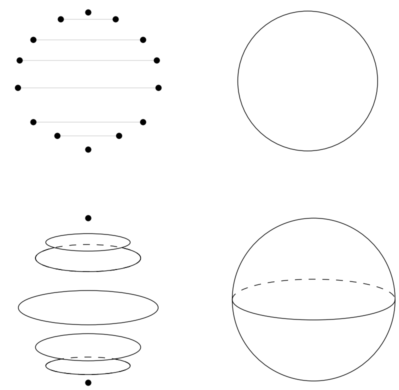

Figure 18: Obtaining $S^n$ as a collection of spheres $S^{n-1}$ between two points.

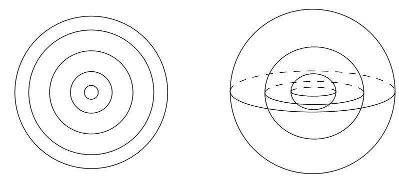

Figure 19: Obtaining $\mathbb{R}^2$ (left) and $\mathbb{R}^3$ (right) as a collection of concentric spheres of all positive radii. In the same way we can decompose $\mathbb{R}^k$ for any $k \in \{1, 2, \ldots\}$.