TDA, Dec. 15 2020

# Stability of persistence diagrams

$S = \{v_1, v_2 \dots v_n\}$ a sample

$S' = \{v_1', v_2', \dots v_n'\}$ perturbed sample

$\{Rips(S,r)\}_r$

$\{Rips(S',r)\}_r$

Rips filtrations

persistence modules (coefficients in a field)

$V = \{H_p(Rips(S,r))\}$ $V' = \{H_p(Rips(S',r)\}$

interleaving distance: $d_i(V, V')$ = smallest $\varepsilon$ for which an $\varepsilon$-interleaving exists

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$D(S)$, $D(S')$ persistence diagrams corresponding to the two Rips filtrations

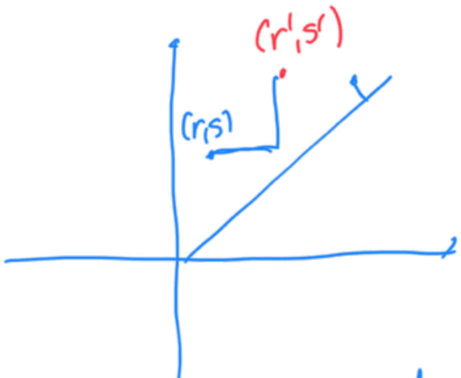
multiset of point $\{a = (r,s)^{\alpha}\}$

\# of generators with this data

birth-death data of a homology generator in the filtration

bottleneck distance

$d_b(D(S), D(S'))$ = minimal cost of matching between $D(S)$ and $D(S')$




$C(M)$ = max distance between pairs in the matching and points outside the matching to the diagonal

$d_\infty((r,s),(r',s')) = \max(|r-r'|, |s-r'|)$

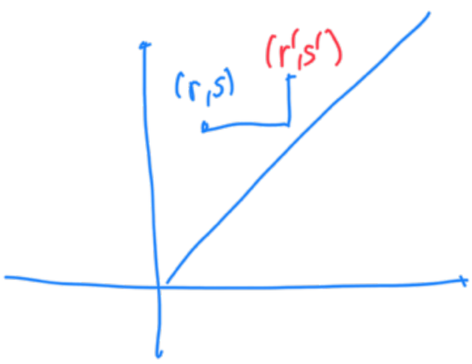
Theorem (stability) $d_b(D(S), D(S')) = d_i(N, N')$

- Sketch of proof: assume that we have an $\varepsilon$-interleaving between $N$ and $N'$. We want to show: $d_b(D(S), D(S')) \le \varepsilon$.

$\gamma$ - birth time $\qquad x \in M \qquad 0$

$$\begin{tikzcd}[column sep=small]
& H_p(\mathrm{Rips}(S,r)) \arrow[r] \arrow[drr] & H_p(\mathrm{Rips}(S,r+\varepsilon)) \arrow[r] \arrow[drr] & H_p(\mathrm{Rips}(S,r+2\varepsilon)) \arrow[r] & \cdots \\
H_p(\mathrm{Rips}(S',r-\varepsilon)) \arrow[r] \arrow[ur] & H_p(\mathrm{Rips}(S',r)) \arrow[r] \arrow[ur] & H_p(\mathrm{Rips}(S',r+\varepsilon)) \arrow[r] \arrow[ur] & {}
\end{tikzcd}$$

$\gamma' \qquad \gamma''$




$\max\{|r-r'|, |s-s'|\} < \varepsilon$

if $\gamma' \neq 0$ then $r-\varepsilon$ is its birth time

(if is appeared at $r-2\varepsilon$, then $\gamma$ would appear before $r$

$\gamma' = 0$, then $\gamma''$ is matched to $\gamma$

↑ birth time is $r+\varepsilon$

$|r-r'| \le \varepsilon$

similar argument $|s-s'| \le \varepsilon$ $\Rightarrow$ if there is a pair $(\gamma, \gamma') \in M$, then the corresponding birth-death points in the persistence diagrams have distance $\le \varepsilon$.

What about $x \in M \Rightarrow d_\infty(x, \Delta) < \varepsilon$

$\Rightarrow d_b(D(S), D(S')) \le \varepsilon$

Recall: $S = \{v_1 \ldots v_n\}$ $S' = \{v_1', \ldots v_n'\}$

if $d(v_i, v_i') \leq \varepsilon$ for all $i$ $\boxed{\text{(an } \varepsilon\text{-perturbation)}}$

then this induces a $2\varepsilon$-interleaving on the Rips filtrations, which induces a $2\varepsilon$-interleaving of the persistence modules

$$V = \{H_p(Rips(S, r))\}_r \quad V' = \{H_p(Rips(S', r))\}_r$$

$\Rightarrow$ by the theorem

$$\boxed{d_b(D(S), D(S')) \leq 2\varepsilon}$$

Stability of Rips persistence diagrams: if the max distance between $S$ and $S'$ is $\varepsilon$, then the bottleneck distance between $D(S)$ and $D(S')$ is $\leq 2\varepsilon$.

{ Rips persistence diagrams are good descriptors of the data set }

Case 2: $f: X \to \mathbb{R}$

the sublevel set filtration:

$$X_a = \{x \in X;\ f(x) \leq a\}$$

$K$ : $K_a = \{\sigma \in K;\ f(v) \leq a$ for all vertices $v \in \sigma\}$

$$f: |K| \to \mathbb{R}$$

If $f'$ is a perturbed map,

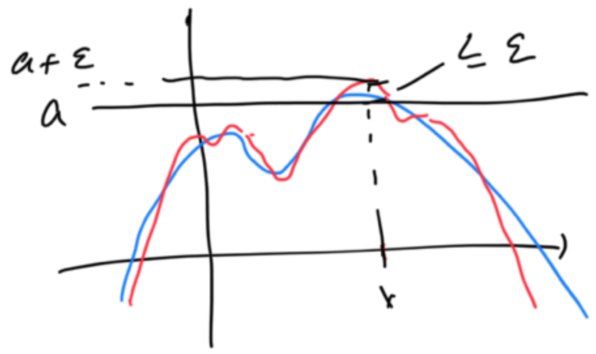
<u>Theorem</u> (stability theorem for maps):

if $\|f - f'\|_\infty = \max_{x \in |K|} \{ |f(x) - f'(x)| \} \leq \varepsilon$, then

$d_b(D(f), D(f')) \leq \varepsilon$.

persistence diagrams of the corresponding sublevel complex filtrations.

Sketch of proof: if $\|f - f'\|_\infty < \varepsilon$, we have an $\varepsilon$-interleaving on the sublevel complex filtrations




a

$K(f)_a \subset K(f')_{a+\varepsilon}$

blue red

this gives an $\varepsilon$-interleaving on the persistence modules which then

implies $d_b(D(f), D(f')) \leq \varepsilon$.

# DISCRETE MORSE THEORY

$K$ simplicial complex (finite)

index the simplices $\sigma \in K$ with a function

$$f: K \to \mathbb{R}$$
$$\sigma \mapsto f(\sigma) \in \mathbb{R}$$

where $f$ has the following properties:

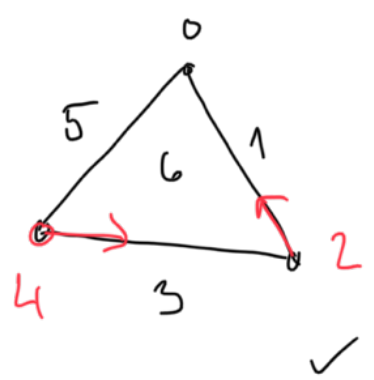
- for any $\sigma^p$, the number $n_1$ of its faces $\tau^{p-1}$ such that $f(\tau) \geq f(\sigma)$ is 0 or 1

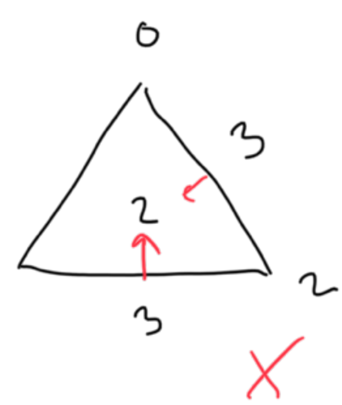
  $n_1 = 0$ or $1$

- for any $\sigma^p$, the number $n_2$ of its cofaces $\mu^{p+1}$ such that $f(\mu) \leq f(\sigma)$ is again 0 or 1

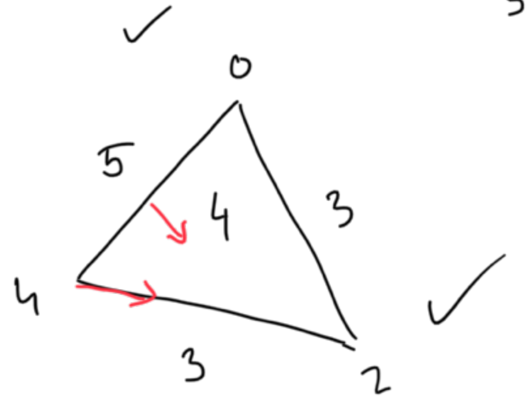
**The function $f$ is a discrete Morse function**

For any $\sigma$, values of $f$ strictly decrease toward its faces, with at most one exception and the values strictly increase towards its cofaces, with at most one exception.




triangle has two exceptions



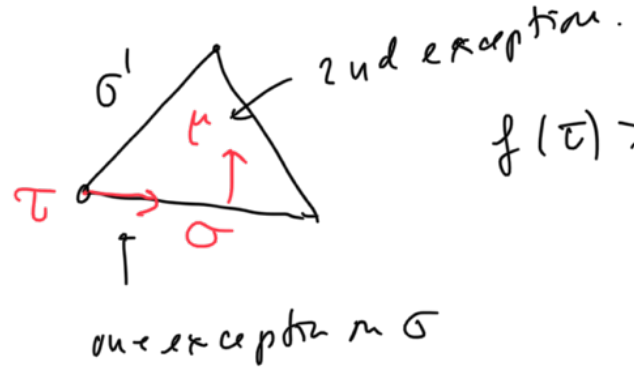

Arrows $\tau^{p-1} \mapsto \sigma^p$ denote the direction of decreasing function values.

If $\sigma^p$ is any simplex in $K$ then only one of the numbers $n_1$ or $n_2$ can be $1$.

So: for any simplex $\sigma$ at most one exception is allowed.

Proof: assume that a simplex $\sigma$ is at the same time a head and a tail of an arrow, i.e.

$\tau^{p-1} \mapsto \sigma^p$ and $\sigma^p \mapsto \mu^{p+1}$ and let $\tau' < \mu^{p+1}$ be a different face from $\tau$. Then:




$$f(\tau) \geq f(\sigma) \geq f(\mu) > f(\sigma') > f(\tau)$$

$\rightarrow\!\!\!\!\times$ a contradiction!

This implies every simplex in $K$ is involved in at most one arrow, either as an arrowhead or as an arrowtail

So, we have a partial matching

$\{\tau \mapsto \sigma\}$ regular pair

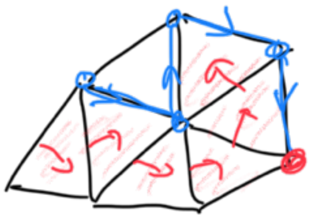
$\{\delta\}$ not in the matching
critical simplex

The collection of arrows $\{\tau \mapsto \sigma\}$ in a partial matching on $K$ is a discrete vector field on $K$.

The simplices of $K$ are subdivided into $\underbrace{\text{critical simplices}}^{C}$, $\underbrace{\text{arrow heads}}_{A}$, $\underbrace{\text{arrow tails}}_{B}$

$K = A \cup B \cup C$

disjoint union



a critical vertex

a 2-path
and a 1-path

a sequence of consecutive arrows is a path in the discrete vector field (forms a path through sx's of two consecutive dimensions in $K$)

$$\tau_1 \mapsto \sigma_1 > \tau_2 \mapsto \sigma_2 > \dots \dots \tau_r \mapsto \sigma_r > \tau_{r+1}$$

A p-path in $K$ consists of consecutive arrows, where $\tau_{i+1}^{p-1} \neq \tau_i^{p-1}$ for every

Function values of $f$ along the path decrease:

$$f(\tau_1) \underset{\text{arrow}}{\geq} f(\sigma_1) \underset{\text{face-relation}}{>} f(\tau_2) \geq f(\sigma_2) > \dots \quad > f(\tau_{r+1})$$

p-paths do not form cycles.
The discrete vector field arising from a discrete Morse function on $K$ is acyclic.

Critical simplices do not appear in a p-path, except maybe at the end, as the ending $(p-1)$-simplex.

Motivation for discrete Morse theory:
critical cells determine the shape of $K$.

A sublevel complex of $K$ corresponding to $a \in \mathbb{R}$:

$$K_a = \{ \sigma \in K;\ f(\sigma) \le a \} \cup \text{all their faces} =$$

$$= \bigcup_{f(\sigma) \le a} \Big( \bigcup_{\tau \le \sigma} \tau \Big)$$

$K$, a value $f$ on every simplex of $K$

$K_a$ - sublevel complexes, form a filtration of $K$.

$a_1 \le a_2 \le a_3 \le \dots \le a_n$ the values of $f$

assume that $f$ is injective (for now)

$a_1 < a_2 < a_3 < \dots < a_n$

sublevel complex filtration

$$K_1 < K_2 < K_3 < \dots < K_n = K$$

$$K_i = K_{a_i}$$

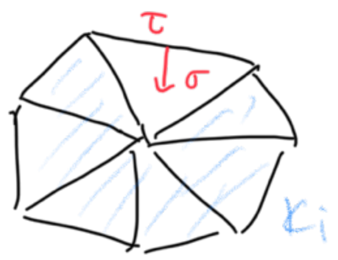
what happens from $K_i$ to $K_{i+1}$.

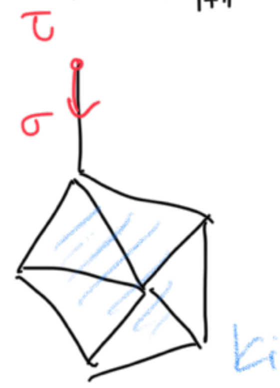
two types of events:

① assume that $a_{i+1} = f(\sigma_{i+1})$ where

where $\sigma_{i+1}$ belong to a pair $\tau^{p-1} \mapsto \sigma^{p}$

so $K_{i+1} = K_i \cup (\sigma^{p}_{i+1} \cup \tau^{p-1})$



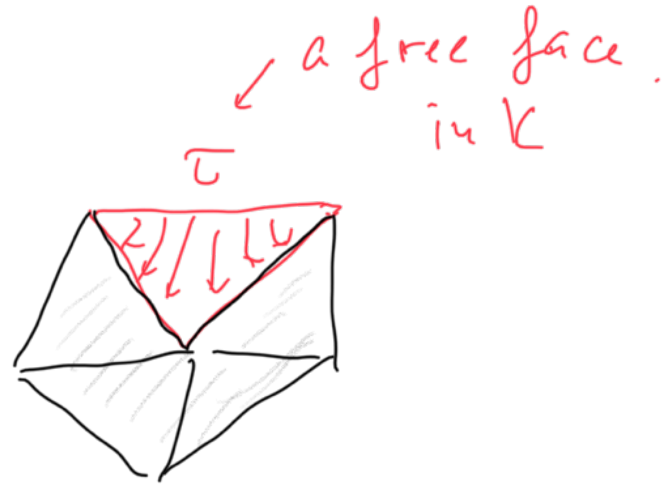

$K_{i+1} = K_i \cup \{\tau \mapsto \sigma\}$

In this case $K_{i+1}$ collapses onto $K_i$ by an elementary collapse: $K_{i+1} \searrow K_i$

An elementary collaps on a simplicial complex $K$ consists of removing a pair of simplices $(\tau^{p-1} < \sigma^{p})$ where $\tau$ is a free face of $\sigma$, that is, it is not the face of any other simplex in $K$.

$K \searrow K - \{\sigma, \tau\}$.

An elementary collaps is a homotopy equivalence: $|K| \simeq |K - \{\sigma, \tau\}|$.




A continuous deformation pushes $\tau$ and $\sigma$ onto the remaining boundary of $\sigma$.

Combinatorially, an elementary collapse corresponds to removing the pair $(\tau < \sigma)$ from $K$.

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$$K_1 \subseteq K_2 \subseteq \dots \subseteq K_i \subseteq K_{i+1} \subseteq \dots \subseteq K_n = K$$

$$K_i = K_{a_i}$$

if $a_{i+1} = f(\sigma^p)$, where $\tau^{p-1} \mapsto \sigma^p$ is a regular pair

then exists an elementary collaps

$$K_{i+1} = K_i \cup \{\tau, \sigma\} \searrow K_i$$

$$K_{i+1} \simeq K_i$$

The next stage $K_{i+1}$ is homotopy equivalent to $K_i$, the homology groups remain the same.

the step from $K_i$ to $K_{i+1}$ is a regular event in the filtration.

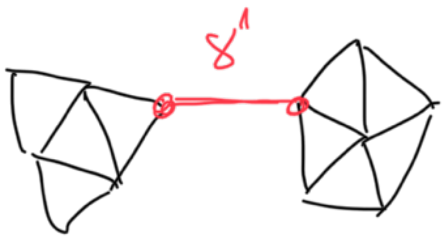
② $K_{i+1} = K_i \cup \gamma$

a critical simplex

$$a_{i+1} = f(\gamma)$$

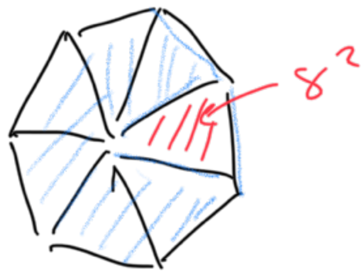
in this case all the faces of $\gamma$ are included already in $K_i$

deaths




$\gamma^p$ kills a homology generator in dimension $p-1$

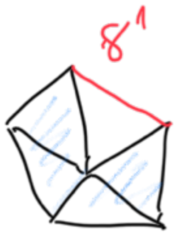
a death



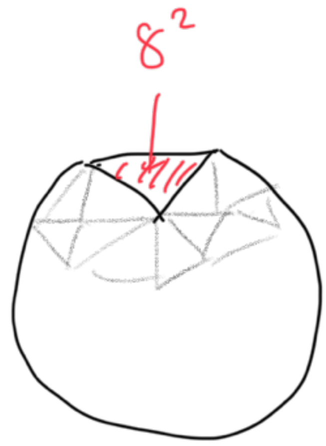

$\gamma^2$ kills a homology generator in dimension 1

a death

births




$\gamma^1$ is the birth of a new 1-dimensional homology generator




$\gamma^2$ form a new 2-dimensional homology generator

Adding a critical simplex, $K_{i+1} = K_i \cup \sigma$ critical

results in a change in the homology from $K_i$ to $K_{i+1}$:

If $\sigma^p$ is a p-dimensional critical simplex, then it either:

① completes a new p-cycle, so

$$\beta_p(K_{i+1}) = \beta_p(K_i)+1 \quad \underline{\text{or}}$$

② forms a new boundary which fills in an existing $(p-1)$-cycle, so

$$\beta_{p-1}(K_{i+1}) = \beta_{p-1}(K_i)-1$$

This tells us: critical simplices affect the shape of $K$.

Special case: K has only one critical simplex

The unique critical simplex $\gamma$ must correspond to the minimum of $f$

since the $f(\gamma) \leq f(\sigma)$ for any coface $\sigma$ of $\gamma$,

$\gamma$ must be a vertex $v$ of $K$.

In the sublevel complex filtration

$\{v_0\} = K_1 \subseteq K_2 \ldots$

all events are regular so there is an elementary collapses $K_{i+1} \searrow K_i$ for all $i$.

So, $K = K_n \simeq K_{n-1} \simeq \ldots \ldots \simeq \{v_0\}$

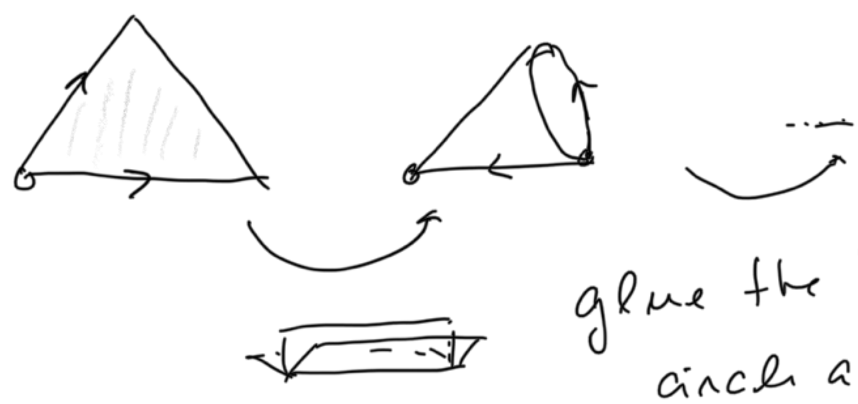
K is contractible.

Definition: A simplicial complex $K$ is collapsible if there exists a sequence of elementary collapses to a point,

$K \searrow \ldots \searrow \{v\}$

Collapsibility is a stronger property than contractibility: there exist contractible simplicial complexes that are not collapsible.

Example: dunce hat



glue the base circle along one of the segments to the vertex

this is contractible but not collapsible: for any triangulation, no free face exists.

Algorithmically collapsibility is much easier to implement than contractibility: as long as a free face exists, a collapse can be done.

# Morse homology of K.

K, discrete Morse function $f$ on K

$C = \{$critical simplices of $f\}$

Recall: to define homology groups we first define the simplicial chain complex

$$0 \to C_k(K) \xrightarrow{\partial_k} C_{k-1}(K) \xrightarrow{\partial_{k-1}} \cdots \to C_1(K) \xrightarrow{\partial_1} C_0(K) \to 0$$

$C_i(K)$ is a abelian group (a vector space if coefficients are $\mathbb{F}$) generated by all $i$-dimensional sx's

$\partial_i : C_i(K) \to C_{i-1}(K)$ boundary homomorphism

$$\sigma = \langle v_0, v_1 \dots v_i \rangle$$

$$\partial_i(\sigma) = \sum_{j=0}^{i} (-1)^j \langle v_0 \dots v_{j-1}, \hat{v}_j, v_{j+1} \dots v_i \rangle$$

Homology groups: $H_i(K) = \dfrac{Z_i(K)}{B_i(K)} = \dfrac{\ker \partial_i}{\operatorname{im} \partial_{i+1}}$

$(\partial \circ \partial = 0)$.

simplicial homology groups

The Morse chain complex: (K, $f: K \to \mathbb{R}$
C : critical sx's of $f$)

K, $f: K \to \mathbb{R}$ Morse function
C critical simplices of $f$.

$$\boxed{0 \longrightarrow M_k(K) \xrightarrow{\partial_k} M_{k-1}(K) \xrightarrow{\partial_{k-1}} \cdots \longrightarrow M_1(K) \xrightarrow{\partial_1} M_0(K) \to 0}$$

where $M_i(K)$, the Morse i-th chain group, is generated by the critical sx's of dimension i

a Morse chain a formal sum

$\sum_j \alpha_j \, \gamma_j^i$ j runs over all critical sx's of dim i.

What are the boundary homomorphisms?