5. Izračunaj vsoto naslednjih geometrijskih vrst.

(a) * $\sum_{n=1}^{\infty} \frac{10}{3^n}$

(b) $\sum_{n=2}^{\infty} \frac{2^n}{3^{2n-1}}$

(c) * $3/2 + 1 + 2/3 + 4/9 + 8/27 + \dots$

(d) $\sum_{n=1}^{\infty} \frac{(-2)^n}{3 \cdot 2^{3n-2}}$

(e) * $\sum_{n=1}^{\infty} (\frac{x}{2})^{3n}$ za tiste $x \in \mathbb{R}$, za katere vrsta konvergira

$$|q| < 1 \Rightarrow$$

- $1 + q + q^2 + \dots + q^k = \frac{1-q^{k+1}}{1-q}$
- $1 + q + q^2 + \dots = \frac{1}{1-q}$
- $q^k + q^{k+1} + q^{k+2} + \dots = \frac{q^k}{1-q}$

(a) * $\sum_{n=1}^{\infty} \frac{10}{3^n} = 10 \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = 10 \cdot \frac{\frac{1}{3}}{1-\frac{1}{3}} = 10 \cdot \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{10}{2} = \underline{\underline{5}}$

$q = \frac{1}{3},\ k = 1$

(b) $\sum_{n=2}^{\infty} \frac{2^n}{3^{2n-1}} = \sum_{n=2}^{\infty} 3 \cdot \frac{2^n}{3^{2n}} = 3 \sum_{n=2}^{\infty} \left(\frac{2}{9}\right)^n = 3 \cdot \frac{\left(\frac{2}{9}\right)^2}{1-\frac{2}{9}} = 3 \cdot \frac{\frac{4}{81}}{\frac{7}{9}} =$

$k = 2$, $q = 2/9$

$= \frac{3 \cdot 4 \cdot 9}{7 \cdot 81} = \frac{4}{7 \cdot 3} = \underline{\underline{\frac{4}{21}}}$

(c) * $3/2 + 1 + 2/3 + 4/9 + 8/27 + \dots = \frac{3}{2} + \frac{3}{2} \cdot \frac{2}{3} + \frac{3}{2} \cdot \left(\frac{2}{3}\right)^2 + \dots = \frac{3}{2}\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \dots\right) = \frac{3}{2} \cdot \frac{1}{1-\frac{2}{3}} =$

$k = 0$, $q = \frac{2}{3}$

$= \frac{3}{2} \cdot \frac{1}{\frac{1}{3}} = \underline{\underline{\frac{9}{2}}}$

(d) $\sum_{n=1}^{\infty} \frac{(-2)^n}{3 \cdot 2^{3n-2}} = \sum_{n=1}^{\infty} \frac{1}{3} \cdot 2^2 \frac{(-2)^n}{2^{3n}} = \frac{4}{3} \sum_{n=1}^{\infty} \left(-\frac{2}{8}\right)^n = \frac{4}{3} \sum_{n=1}^{\infty} \left(-\frac{1}{4}\right)^n = \frac{4}{3} \frac{\left(-\frac{1}{4}\right)}{1-\left(-\frac{1}{4}\right)} = \frac{4}{3} \frac{-\frac{1}{4}}{\frac{5}{4}} = \frac{4}{3} \cdot \left(-\frac{1}{5}\right) = \underline{\underline{-\frac{4}{15}}}$

$q = -\frac{1}{4},\ k = 1$

(e) * $\sum_{n=1}^{\infty} (\frac{x}{2})^{3n}$, za tiste $x \in \mathbb{R}$, za katere vrsta konvergira.

$\sum_{n=1}^{\infty} (\frac{x}{2})^{3n} = \sum_{n=1}^{\infty} \left(\left(\frac{x}{2}\right)^3\right)^n = \sum_{n=1}^{\infty} \left(\frac{x^3}{8}\right)^n = \frac{\frac{x^3}{8}}{1-\frac{x^3}{8}} = \frac{\frac{x^3}{8}}{\frac{8-x^3}{8}} = \underline{\underline{\frac{x^3}{8-x^3}}}$

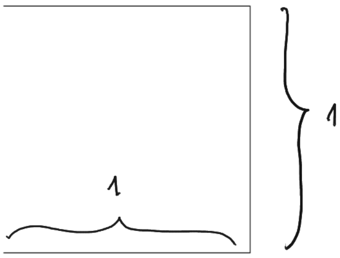
za $q = \frac{x^3}{8}$, $k = 1$, če je $\left|\frac{x^3}{8}\right| < 1$

$$\frac{|x^3|}{8} < 1$$
$$|x^3| < 8$$
$$|x|^3 < 8$$
$$\underline{\underline{|x| < 2}}$$

Vsota je $\frac{x^3}{8-x^3}$ za $x \in (-2, 2)$. Za ostale $x$ vrsta ne konvergira.

6. Izračunaj obseg Hilbertove krivulje:

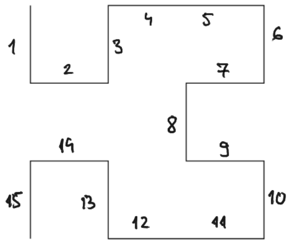
$n = 1$




3 stranic dolžine 1 $\rightsquigarrow$ $\sigma_1 = 3$ $\quad 2+1$

$4-1 \qquad \frac{1}{2-1}$

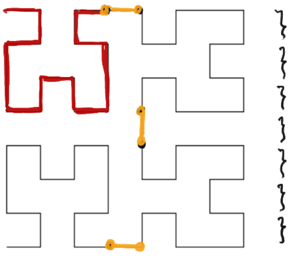
$n = 2$




15 stranic dolžine $\frac{1}{3}$ $\rightsquigarrow$ $\sigma_2 = 15 \cdot \frac{1}{3} = 5$ $\quad 4+1$

$16-1 \qquad \frac{1}{4-1}$

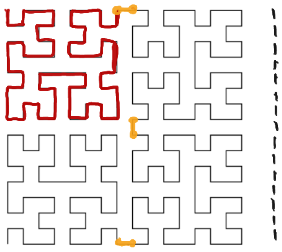
$n = 3$



$3 + 4 \cdot 15 = 63$ stranic dolžine $\frac{1}{7}$ $\rightsquigarrow$ $\sigma_3 = 63 \cdot \frac{1}{7} = 9$ $\quad 8+1$

$64-1 \qquad \frac{1}{8-1}$

$n = 4$



$3 + 4 \cdot 63 = 255$ stranic dolžine $\frac{1}{15}$ $\rightsquigarrow$ $\sigma_4 = 255 \cdot \frac{1}{15} = 17$ $\quad 16+1$

$256-1 \qquad \frac{1}{16-1}$

$n=1 \quad n=2 \quad n=3 \quad n=4$

$3, 15, 63, 255, \ldots$ $(4^n - 1)$ stranic ✓

$n=1 \quad n=2 \quad n=3 \quad n=4$

$1, \frac{1}{3}, \frac{1}{7}, \frac{1}{15} \cdots$ $\left(\frac{1}{2^n-1}\right)$ dolžina ✓

$$\sigma_n = (4^n - 1) \cdot \frac{1}{2^n - 1}$$

$$\sigma_n = \frac{4^n - 1}{2^n - 1} = \frac{(2^n)^2 - 1}{2^n - 1} = \frac{(2^n-1)(2^n+1)}{2^n-1} = 2^n + 1 \; ✓$$

$$\sigma = \lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} (2^n + 1) = \infty$$

★ Na vsakem koraku se prejšnji vzorec ponovi v 4 pomanjšanih kopijah, povezanih s 3 dodatnimi robovi.

$s_n$ … število stranic na $n$-tem koraku

$d_n$ … dolžina stranice na $n$-tem koraku

$s_{n+1} = 4 \cdot s_n + 3$, $s_1 = 3$, z indukcijo dokaži: $s_n = 4^n - 1$

$d_{n+1} = \frac{1}{\frac{2}{d_n} + 1}$, $d_1 = 1$, z indukcijo dokaži: $d_n = \frac{1}{2^n - 1}$

$+1 \rightarrow$ št. robov v navpični smeri

Baza: $s_1 = 4^1 - 1$

$3 = 3$ ✓

Ind. predp. $s_n = 4^n - 1$

$s_{n+1} = 4 \cdot s_n + 3 = 4(4^n - 1) + 3 = 4^{n+1} - 4 + 3 = 4^{n+1} - 1$ ✓

Baza: $d_1 = \frac{1}{2^1 - 1}$

$1 = \frac{1}{2-1}$ ✓

Ind. predp.: $d_n = \frac{1}{2^n - 1}$

Ind. korak: $d_{n+1} = \frac{1}{\frac{2}{d_n} + 1} = \frac{1}{\frac{2}{\frac{1}{2^n-1}} + 1} = \frac{1}{2(2^n-1)+1} = \frac{1}{2^{n+1} - 1}$ ✓

# Osnove matematične analize
## Vaje 5

1. Z uporabo korenskega, kvocientnega ali primerjalnega kriterija ugotovi, katere od spodnjih vrst kovergirajo in katere ne.

(a) $\sum_{n=1}^{\infty} \frac{n}{2^n}$, (c) $\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(3n)!}$, (e) $\sum_{n=1}^{\infty} \frac{n^3 \cdot 2^{3n}}{n^4+1}$, (g) $\sum_{n=5}^{\infty} \frac{1}{n!}$.

(b) $\sum_{n=1}^{\infty} \frac{n^3}{n^5+3}$, (d) $\sum_{n=1}^{\infty} \frac{3^n}{4^n+4}$, (f) $\sum_{n=1}^{\infty} \frac{n!}{\pi^n}$,

**KVOCIENTNI KRITERIJ (d'Alembertov kriterij)**

$$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

- $r < 1 \Rightarrow$ vrsta absolutno konvergira
- $r > 1 \Rightarrow$ vrsta divergira
- $r = 1 \Rightarrow$ ???

**KORENSKI KRITERIJ (Cauchyjev kriterij)**

$$r = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

- $r < 1 \Rightarrow$ vrsta absolutno konvergira
- $r > 1 \Rightarrow$ vrsta divergira
- $r = 1 \Rightarrow$ ???

**PRIMERJALNI KRITERIJ**

$\sum b_n$ absolutno konvergentna in $|a_n| < |b_n|$ za vse $n \geq n_0$

$\Rightarrow \sum a_n$ je absolutno konvergentna

(a) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

- $$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \frac{2^n(n+1)}{2^{n+1}n} = \frac{n+1}{2n}$$

$$r = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n+1}{2n} = \underline{\underline{\frac{1}{2}}}$$

$r < 1 \Rightarrow$ po kvocientnem kriteriju $\underline{\underline{\text{konvergira}}}$

- 2. možnost:

$$\boxed{\lim_{n\to\infty} \sqrt[n]{n} = 1}$$

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{n}{2^n}} = \frac{\sqrt[n]{n}}{2}$$

$$r = \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{\sqrt[n]{n}}{2} = \frac{1}{2}\lim_{n\to\infty}\sqrt[n]{n} = \underline{\underline{\frac{1}{2}}}$$

$r < 1 \Rightarrow$ po korenskem kriteriju $\underline{\underline{\text{konvergira}}}$

(b) $\sum_{n=1}^{\infty} \frac{n^3}{n^5+3}$ $\oplus$, ne rabimo $|\cdot|$

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \quad (\text{Euler, } 1735)}$$

$$a_n = \frac{n^3}{n^5+3} \qquad b_n = \frac{1}{n^2}$$

$$b_n - a_n = \frac{1}{n^2} - \frac{n^3}{n^5+3} = \frac{n^5+3-n^5}{n^2(n^5+3)} = \frac{3}{n^2(n^5+3)} > 0$$

$\Rightarrow a_n < b_n$ za vse $n$

$\sum b_n$ konvergira $\Rightarrow \sum a_n$ $\underline{\underline{\text{konvergira}}}$ (primerjalni kriterij)

(c) $\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(3n)!}$

$$\frac{a_{n+1}}{a_n} = \left(\frac{\frac{(n+1)!\,(n+2)!}{(3n+3)!}}{\frac{n!\,(n+1)!}{(3n)!}}\right) = \frac{\cancel{(n+1)!}\,\cancel{n!}\,(n+1)(n+2)\cancel{(3n)!}}{\cancel{n!}\,\cancel{(n+1)!}\,\cancel{(3n)!}\,(3n+1)(3n+2)(3n+3)} = \frac{\cancel{(n+1)}(n+2)}{(3n+1)(3n+2)\cdot 3\cdot\cancel{(n+1)}} = \frac{n+2}{3(3n+1)(3n+2)}$$

$$r = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n+2}{3(3n+1)(3n+2)} = \lim_{n\to\infty}\frac{\frac{1}{n}+\frac{2}{n^2}}{3(3+\frac{1}{n})(3+\frac{2}{n})} = \underline{\underline{0}}$$

$\frac{1}{n}+\frac{2}{n^2} \to 0$, $3(3+\frac{1}{n})(3+\frac{2}{n}) \to 27$

$\rightsquigarrow r < 1 \Rightarrow$ po kvocientnem kriteriju $\underline{\underline{\text{konvergira}}}$

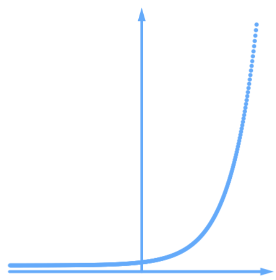
(d) $\sum_{n=1}^{\infty} \frac{3^n}{4^n+4}$ $\oplus$ $\oplus$ $\to$ $|\cdot|$ ignoriramo

$b_n = \frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n$ je geometrijska vrsta s $q = \frac{3}{4}$, zato konvergira

$$b_n - a_n = \frac{3^n}{4^n} - \frac{3^n}{4^n+4} = \frac{3^n(4^n+4) - 3^n\cdot 4^n}{4^n(4^n+4)} = \frac{3^n\cdot 4^n + 3^n\cdot 4 - 3^n\cdot 4^n}{4^n(4^n+4)} = \frac{3^n \cdot 4}{4^n(4^n+4)} > 0$$

$\Rightarrow a_n < b_n$ za vse $n$

$\sum b_n$ konvergira $\Rightarrow \sum a_n$ $\underline{\underline{\text{konvergira}}}$ (primerjalni kriterij)



$a^x > 0$ za $a > 0$

(e) $\sum_{n=1}^{\infty} \frac{n^3 \cdot 2^{3n}}{n^4+1}$

$2^{3n+3} = 2^{3n} \cdot 2^3$

- $$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^3 \cdot 2^{3(n+1)}}{(n+1)^4+1}}{\frac{n^3 \cdot 2^{3n}}{n^4+1}} = \frac{(n+1)^3 \cdot 2^{3(n+1)} \cdot (n^4+1)}{n^3 \cdot 2^{3n} \cdot ((n+1)^4+1)} = \frac{(n+1)^3 \cdot 8 \cdot (n^4+1)}{n^3 \cdot ((n+1)^4+1)} \xrightarrow{n\to\infty} 8 > 1$$

$r = 8 > 1 \Rightarrow$ vrsta divergira (kvocientni kriterij)

- 2. možnost

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{n^3 \cdot 2^{3n}}{n^4+1}} = 8\sqrt[n]{\frac{n^3}{n^4+1}} \overset{/:n^3}{=} 8\sqrt[n]{\frac{1}{n+\frac{1}{n^3}}} = \frac{8}{\sqrt[n]{n+\frac{1}{n^3}}}$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty}\left(\frac{8}{\underbrace{\sqrt[n]{n}}_{\to 1}+\underbrace{\tfrac{1}{n^3}}_{0}}\right) = 8 \rightsquigarrow r = 8 > 1 \Rightarrow \text{vrsta divergira (korenski kriterij)}$$

(f) $\sum_{n=1}^{\infty} \frac{n!}{\pi^n}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{\pi^{n+1}}}{\frac{n!}{\pi^n}} = \frac{(n+1)! \cdot \pi^n}{\pi^{n+1} \cdot n!} = \frac{n+1}{\pi}$$

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{\pi} = \infty > 1 \Rightarrow \text{vrsta divergira (kvocientni kriterij)}$$

(g) $\sum_{n=5}^{\infty} \frac{1}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$$

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1 \Rightarrow \text{po kvocientnem kriteriju konvergira}$$

2. Za naslednje vrste ugotovi, če konvergirajo absolutno ali pogojno, ali divergirajo.

(a) $\sum_{n=1}^{\infty}(-1)^n\frac{1}{1+\frac{1}{n}}$, (b) $\sum_{n=2}^{\infty}(-1)^n\frac{1}{n\log n}$, (c) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$.

$\sum a_n$ konvergira absolutno $\overset{\text{def.}}{\Leftrightarrow}$ $\sum |a_n|$ konvergira

$\sum a_n$ konvergira pogojno, če konvergira, ampak ne konvergira absolutno

$\sum a_n$ konvergira absolutno $\Rightarrow$ $\sum a_n$ konvergira

(a) $\sum_{n=1}^{\infty}(-1)^n\frac{1}{1+\frac{1}{n}}$

- $a_n = \frac{1}{1+\frac{1}{n}} = \frac{1}{\frac{n+1}{n}} = \frac{n}{n+1}$

$\lim\limits_{n\to\infty} a_n = 1 \Rightarrow \sum a_n$ ni konvergentna

$\Rightarrow \sum(-1)^n a_n$ ni absolutno konvergentna

- Leibnizov kriterij ne pove ničesar ($a_n \not\to 0$).
- $b_n = (-1)^n a_n = (-1)^n\frac{n}{n+1}$

$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty}(-1)^n\underbrace{\frac{n}{n+1}}_{\to 1} = \lim\limits_{n\to\infty}(-1)^n$ ne obstaja (dve stekališči, 1 in −1)

Ker členi ne gredo proti 0, $\sum b_n$ ne konvergira.

$\Rightarrow \sum(-1)^n a_n$ ne konvergira niti absolutno niti pogojno.

LEIBNIZOV KRITERIJ

$\lim\limits_{n\to\infty} a_n = 0$ in za vse $n$ $a_{n+1} < a_n$

$\Rightarrow \sum_{n=1}^{\infty}(-1)^n a_n$ je konvergentna

(b) $\sum_{n=2}^{\infty}(-1)^n\frac{1}{n\log n}$

$a_n = \frac{1}{n\log n}$

$\lim\limits_{n\to\infty} a_n = 0$

$b_n = n\log n$

$b_{n+1} - b_n = (n+1)\log(n+1) - n\log n = \log(n+1)^{n+1} - \log n^n = \log\frac{(n+1)^{n+1}}{n^n} = \log\Big(\underbrace{\left(\frac{n+1}{n}\right)^n}_{\left(1+\frac{1}{n}\right)^n\to e}\cdot\underbrace{(n+1)}_{\to\infty}\Big) \xrightarrow{n\to\infty} \infty$

$\Rightarrow b_{n+1} - b_n > 0$ za vse $n > n_0$ $\Rightarrow \{b_n\}$ je naraščajoče $\Rightarrow \{a_n\}$ je padajoče

HIPERHARMONIČNA VRSTA

$\sum\frac{1}{n^k}$ konvergira $\Leftrightarrow k > 1$ $(k\in\mathbb{R})$

$\Rightarrow$ Po Leibnizovem kriteriju je $\sum(-1)^n a_n$ konvergentna

Ali je absolutno konvergentna? $\sum a_n < ?$

- $\frac{a_{n+1}}{a_n} = \frac{n\log n}{(n+1)\log(n+1)} \xrightarrow{n\to\infty} 1$ $\Rightarrow$ kvocientni kriterij ne pove ničesar
- primerjalni kriterij: $a_n = \frac{1}{n\log n} > c_n = ?$, kjer $\sum c_n$ divergira

PRIMERJALNI KRITERIJ 2

$\sum b_n$ absolutno divergentna in $|a_n| > |b_n|$ za vse $n \geq n_0$

$\Rightarrow \sum a_n$ je absolutno divergentna

$$\sum_{n=2}^{\infty} a_n = \frac{1}{2\log 2} + \frac{1}{3\log 3} + \frac{1}{4\log 4} + \frac{1}{5\log 5} + \frac{1}{6\log 6} + \frac{1}{7\log 7} + \frac{1}{8\log 8} + \frac{1}{9\log 9} + \dots \overset{*}{\geq}$$

$$\overset{*}{\geq} \underbrace{\frac{1}{2\log 2}}_{2^0\times} + \underbrace{\frac{1}{4\log 4} + \frac{1}{4\log 4}}_{2^1\times} + \underbrace{\frac{1}{8\log 8} + \frac{1}{8\log 8} + \frac{1}{8\log 8} + \frac{1}{8\log 8}}_{2^2\times} + \underbrace{\frac{1}{16\log 16} + \dots + \frac{1}{16\log 16}}_{2^3\times} + \dots = \sum_{n=2}^{\infty}\frac{2^{n-1}}{2^n\log 2^n} =$$

$$= \frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{\log 2^n} = \frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n\log 2} = \frac{1}{2\log 2}\underbrace{\sum_{n=2}^{\infty}\frac{1}{n}}_{\text{divergira}} > \infty$$

$* \; n < m \Rightarrow \log n < \log m \Rightarrow n\log n < m\log m \Rightarrow \frac{1}{n\log n} > \frac{1}{m\log m}$

$\Rightarrow \sum_{n=2}^{\infty} a_n$ divergira po primerjalnem kriteriju $\Rightarrow \sum(-1)^n a_n$ ni absolutno konvergentna.

(c) $\underbrace{\sum_{n=1}^{\infty}(-1)^{n+1}\underbrace{\frac{1}{\sqrt{n}}}_{a_n}}_{b_n}$ $\lim\limits_{n\to\infty} a_n = 0$ $\lim\limits_{n\to\infty} b_n = 0$ → po Leibnizovem kriteriju je $\sum(-1)^n a_n$ konvergentna

$$a_{n+1} - a_n = \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}} = \frac{\sqrt{n}-\sqrt{n+1}^{\ominus}}{\sqrt{n(n+1)}_{\oplus}} < 0$$

Ali je absolutno konvergentna? (tj. $\sum a_n < \infty$ ?)

- $\left|\frac{a_{n+1}}{a_n}\right| = \frac{\sqrt{n}}{\sqrt{n+1}} = \sqrt{\frac{n}{n+1}}$

  $r = \lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\sqrt{\frac{n}{n+1}} = 1$ ⇒ kvocientni kriterij ne pove ničesar

$$\boxed{\lim_{n\to\infty}\sqrt[n]{n} = 1}$$

- $\sqrt[n]{a_n} = \sqrt[n]{\frac{1}{\sqrt{n}}} = \left(n^{-\frac{1}{2}}\right)^{\frac{1}{n}} = \left(n^{\frac{1}{n}}\right)^{-\frac{1}{2}} = \left(\underbrace{\sqrt[n]{n}}_{\to 1}\right)^{-\frac{1}{n}} \xrightarrow{n\to\infty} 1$

  ⇒ korenski kriterij ne pove ničesar.

- $\sum a_n = \sum \frac{1}{\sqrt{n}}$ , $\sum c_n = \sum \frac{1}{n}$

  $|a_n| > |c_n|$ za vse $n$, $\sum|c_n|$ je divergentna ⇒ $\sum|a_n|$ je divergentna po primerjalnem kriteriju

  ⇒ $\sum(-1)^n a_n$ $\boxed{\text{NI}}$ absolutno konvergentna

- 2. način:

$$r_n = n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\sqrt{\frac{n+1}{n}} - 1\right) = n\sqrt{\frac{n+1}{n}} - n = \sqrt{n^2+n} - n$$

$$r = \lim_{n\to\infty} r_n = \lim_{n\to\infty}\left(\sqrt{n^2+n} - n\right)^{\infty-\infty} = \lim_{n\to\infty}\frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \lim_{n\to\infty}\frac{n \;{:n}}{\sqrt{n^2+n}+n \;{:n}} = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}$$

$r < 1$ ⇒ po Raabejevem kriteriju $\sum a_n$ divergira ⇒ $\sum(-1)^n a_n$ $\boxed{\text{NI}}$ absolutno konvergentna

RAABEJEV KRITERIJ

$a_n > 0$ , $r_n = n\left(\frac{a_n}{a_{n+1}} - 1\right)$ , $r = \lim\limits_{n\to\infty} r_n$

- $r < 1$ ⇒ $\sum a_n$ divergira
- $r > 1$ ⇒ $\sum a_n$ konvergira } !
- $r = 1$ ⇒ ???

} to ni napaka, pogoj je ravno nasproten kot pri kvocientnem in korenskem

3. Ugotovi, za katere $x$ vrsta

$$\sum_{n=0}^{\infty}\left(-\frac{2}{x}\right)^n$$

konvergira in izračunaj njeno vsoto.

Geometrijska vrsta s $q = -\frac{2}{x}$ konvergira za $|q| < 1$

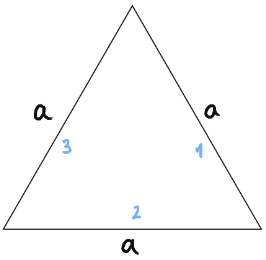
$|q| < 1 \Rightarrow \left|-\frac{2}{x}\right| < 1 \Rightarrow \frac{2}{|x|} < 1 \Rightarrow 2 < |x|$

$\Rightarrow$ Konvergira za $x < -2$ in $x > 2$ oz. $x \in (-\infty, -2) \cup (2, \infty)$.

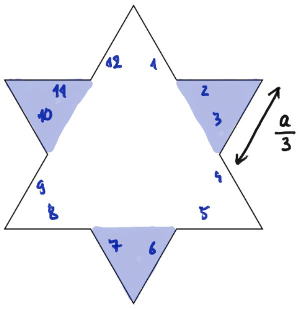
Vsota: $\sum_{n=0}^{\infty}\left(-\frac{2}{x}\right)^n = \sum_{n=0}^{\infty} q^n = \frac{1}{1-q} = \frac{1}{1-\left(-\frac{2}{x}\right)} = \frac{1}{1+\frac{2}{x}} = \frac{1}{\frac{x+2}{x}} = \underline{\underline{\frac{x}{x+2}}}$ (za $|x| > 2$)

4. Kochova snežinka je fraktal, ki ga dobimo z zaporedjem iteracij kot na spodnji sliki.

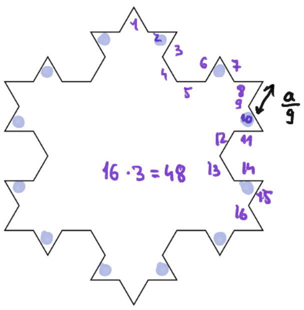
Recimo, da pri $n = 0$ začnemo z enakostraničnim trikotnikom s stranico $a$. Poišči geometrijski vrsti, ki določata ploščino in obseg Kochove snežinke. Seštej jih. Kolikšni sta ploščina in obseg izraženi z $a$?



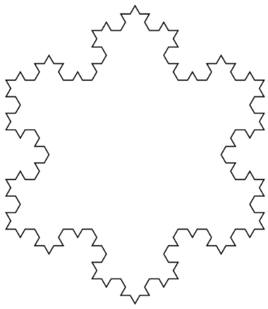

$n = 0$



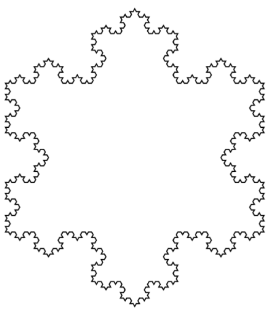

$n = 1$




$n = 2$



$n = 3$



$n = 4$

$\sigma_0 = 3a$

$\sigma_1 = 12 \cdot \frac{a}{3} = 4a$

$\sigma_2 = 48 \cdot \frac{a}{9} = \frac{16}{3} a$

$\vdots$

$\sigma_n = s_n \cdot d_n$

$d_n$: dolžina stranic na $n$-tem koraku

$s_n$: število stranic na $n$-tem koraku

$$\left.\begin{aligned} s_n &= 4 s_{n-1} \\ s_0 &= 3 \end{aligned}\right\} s_n = 3 \cdot 4^n$$

$$\left.\begin{aligned} d_n &= \frac{d_{n-1}}{3} \\ d_0 &= a \end{aligned}\right\} d_n = \frac{a}{3^n}$$

$$\Rightarrow \sigma_n = 3 \cdot 4^n \cdot \frac{a}{3^n} = \frac{3}{a} \cdot \left(\frac{4}{3}\right)^n$$

$$\sigma = \lim_{n \to \infty} \sigma_n = \lim_{n \to \infty} \left( \frac{3}{a} \cdot \left(\frac{4}{3}\right)^n \right) =$$

$$= \frac{3}{a} \lim_{n \to \infty} \left(\frac{4}{3}\right)^n = \infty$$

$$\underline{\underline{\sigma = \infty}}$$

$p_0 = \frac{a^2\sqrt{3}}{4}$

$p_1 = p_0 + 3 \cdot \frac{\left(\frac{a}{3}\right)^2 \sqrt{3}}{4}$

$p_2 = p_1 + 12 \cdot \frac{\left(\frac{a}{9}\right)^2 \sqrt{3}}{4}$

$\vdots$

$$p_n = p_{n-1} + s_{n-1} \cdot \frac{(d_n)^2 \sqrt{3}}{4} =$$

$$= p_{n-1} + 3 \cdot 4^{n-1} \frac{\frac{a^2}{3^{2n}} \sqrt{3}}{4} =$$

$$= p_{n-1} + \frac{3}{4} a^2 \sqrt{3} \, \frac{4^{n-1}}{9^n} = \qquad 4^{n-1} \to \frac{1}{4} \cdot 4^n$$

$$= p_{n-1} + \frac{3a^2\sqrt{3}}{16} \left(\frac{4}{9}\right)^n$$

$n=1$, $n=2$

$$p = \frac{a^2\sqrt{3}}{4} + 3 \cdot \frac{\left(\frac{a}{3}\right)^2 \sqrt{3}}{4} + 12 \cdot \frac{\left(\frac{a}{9}\right)^2 \sqrt{3}}{4} + \dots$$

$$= \frac{a^2\sqrt{3}}{4} + \sum_{n=1}^{\infty} \frac{3a^2\sqrt{3}}{16} \left(\frac{4}{9}\right)^n =$$

$$= \frac{a^2\sqrt{3}}{4} + \frac{3a^2\sqrt{3}}{16} \sum_{n=1}^{\infty} \left(\frac{4}{9}\right)^n =$$

$$= \frac{a^2\sqrt{3}}{4} + \frac{3a^2\sqrt{3}}{16} \, \frac{\frac{4}{9}}{1 - \frac{4}{9}} =$$

$$= \frac{a^2\sqrt{3}}{4} + \frac{3a^2\sqrt{3}}{16} \, \frac{\frac{4}{9}}{\frac{5}{9}} =$$

$$= \frac{a^2\sqrt{3}}{4} + \frac{3a^2\sqrt{3}}{16} \, \frac{4}{5} =$$

$$= \frac{a^2\sqrt{3}}{4} + \frac{3a^2\sqrt{3}}{4} \, \frac{1}{5} =$$

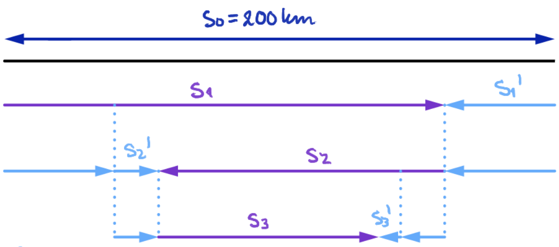
$$= \frac{a^2\sqrt{3}}{4} \left(1 + \frac{3}{5}\right) = \frac{a^2\sqrt{3}}{4} \cdot \frac{8}{5} =$$

$$= \frac{2a^2\sqrt{3}}{5}$$

$$\underline{\underline{p = \frac{2a^2\sqrt{3}}{5}}}$$

5. Dva vlaka na istem tiru potujeta eden proti drugemu s hitrostjo 100km/h. Supermuha zapusti prvi vlak s hitrostjo 150km/h, odleti proti drugemu, se v trenutku obrne in z enako hitrostjo spet odleti proti prvemu. Tam se spet obrne in odleti proti drugemu vlaku ... Ugotovi kolikšno razdaljo bo preletela Supermuha do trenutka, ko se vlaka zaletita. Na začetku sta bila vlaka oddaljena 200km.

Rešitev: Supermuha prepotuje razdaljo 150km.




$$s_1 + s_1' = s_0$$
$$s_2 + s_2' = s_0 - 2s_1' = s_1 + s_1' - 2s_1' = s_1 - s_1'$$
$$s_3 + s_3' = s_0 - 2s_1' - 2s_2' = s_2 + s_2' - 2s_2' = s_2 - s_2'$$
$$s_4 + s_4' = s_3 - s_3'$$
$$\vdots$$
$$s_n + s_n' = s_{n-1} - s_{n-1}'$$

$t = \frac{s}{v}$

$$t_n = \frac{s_n}{v_m} = \frac{s_n'}{v_v} \rightsquigarrow s_n' = \frac{v_v}{v_m} s_n = \frac{100\,km/h}{150\,km/h} s_n = \frac{2}{3} s_n$$

- $s_1 + s_1' = s_1 + \frac{2}{3}s_1 = \frac{5}{3}s_1 = s_0 \rightsquigarrow \underset{\sim}{s_1} = \frac{3}{5}s_0 = \underset{\sim}{120\,km} \quad (s_1' = 80\,km)$
- $s_2 + s_2' = s_0 - 2s_1' = 200\,km - 2\cdot 80\,km = 40\,km$

  $s_2 + s_2' = \frac{5}{3}s_2 = 40\,km \rightsquigarrow \underset{\sim}{s_2 = 24\,km}$
- $s_n + s_n' = s_{n-1} - s_{n-1}'$

  $s_n + \frac{2}{3}s_n = s_{n-1} - \frac{2}{3}s_{n-1}$

  $\frac{5}{3}s_n = \frac{1}{3}s_{n-1}$

  $\underline{\underline{s_n = \frac{1}{5}s_{n-1}}} \rightsquigarrow s = s_1 + s_2 + s_3 + s_4 + \dots = s_1 + \frac{1}{5}s_1 + \left(\frac{1}{5}\right)^2 s_1 + \dots = s_1 \sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^n = s_1 \cdot \frac{1}{1-\frac{1}{5}} =$

  $= s_1 \frac{1}{\frac{4}{5}} = \frac{5}{4}s_1 = \frac{5}{4}\cdot 120\,km = \underline{\underline{150\,km}}$

<u>2. način</u>: $s_n$ = pot muhe na $n$-tem koraku

$s_n'$ = pot vlaka na $n$-tem koraku

$v_m$ = hitrost muhe = 150 km/h

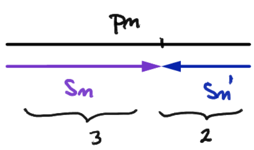
$v_v$ = hitrost vlaka = 100 km/h

$s_n : s_n' = v_m : v_v = 150\,km/h : 100\,km/h = 3:2 \Rightarrow \frac{s_n}{s_n'} = \frac{3}{2} \Rightarrow s_n' = \frac{2}{3}s_n$

$p_n$ = pot na $n$-tem koraku, $p_1 = 200\,km$

$p_n = p_{n-1} - 2s_{n-1}' = p_{n-1} - 2\cdot\frac{2}{3}s_{n-1} = p_{n-1} - \frac{4}{3}s_{n-1} =$

$= p_{n-1} - \frac{4}{3}\cdot\frac{3}{5}p_{n-1} = p_{n-1} - \frac{4}{5}p_{n-1} = \frac{1}{5}p_{n-1}$




$s_n : s_n' = 3:2$

$\Rightarrow s_n = \frac{3}{5}p_n$

$\Rightarrow$ Na vsakem naslednjem koraku je celotna pot 5× krajša $\Rightarrow$ pot supermuhe je 5× krajša $(s_n = \frac{1}{5}s_{n-1})$.

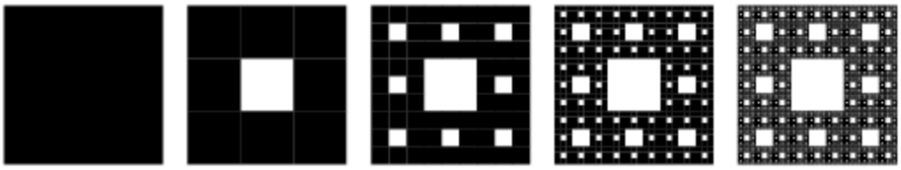
Prva pot supermuhe je $s_1 = \frac{3}{5}p_1 = \frac{3}{5}\cdot 200\,km = 120\,km$, naslednja $s_2 = \frac{1}{5}\cdot 120\,km = 24\,km$,

$s_3 = \frac{1}{5}\cdot 24\,km = 4.8\,km, \dots$

$$s = s_1 + s_2 + s_3 + \dots = s_1 + \frac{1}{5}s_1 + \left(\frac{1}{5}\right)^2 s_1 + \dots = s_1 \sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^n = s_1 \cdot \frac{1}{1-\frac{1}{5}} = \frac{5}{4}s_1 = \frac{5}{4}\cdot 120\,km = 150\,km.$$

<u>3. način</u>: Vlaka potujeta z 100 km/h. Po 1h naredi vsak 100 od začetnih 200 km in se zaletita. V tej 1h je supermuha prepotovala $s = 1h \cdot 150\,km/h = 150\,km$.

6. Izračunaj obseg in ploščino preproge Sierpinskega.



Rešitev: Obseg je neskončen, ploščina pa 0.

$p_0 = a^2$

$p_1 = \frac{8}{9} a^2$

$\vdots$

$p_n = \left(\frac{8}{9}\right)^n a^2$

$$p = \lim_{n\to\infty} p_n = a^2 \lim_{n\to\infty} \overbrace{\left(\frac{8}{9}\right)^n}^{\longrightarrow 0} = 0$$

$\sigma_0 = 4a$

$\sigma_1 = \sigma_0 + 4\cdot\frac{a}{3}$

$\sigma_2 = \sigma_1 + 8\cdot 4\cdot\frac{a}{9}$

$\vdots$

$\sigma_n = \sigma_{n-1} + k_n \cdot 4 \cdot \frac{a}{3^n}$

$\downarrow$

število kvadratov, ki jih izrežemo na $n$-tem koraku

$k_n = 8^{n-1}$

$$\sigma_n = \sigma_{n-1} + 8^{n-1}\cdot 4\cdot\frac{a}{3^n} = \sigma_{n-1} + \left(\frac{8}{3}\right)^n \cdot \frac{4}{8} a$$

$$\sigma_n = \sigma_{n-1} + \frac{a}{2}\cdot \underbrace{\left(\frac{8}{3}\right)^n}_{\xrightarrow[n\to\infty]{}\infty}, \qquad \sigma = \lim_{n\to\infty} \sigma_n$$

$$\sigma = \infty$$