1. Denoting $\mathbf{x} = [x, y]^{\mathsf{T}}$ find the general solutions to the system of differential equations $\dot{\mathbf{x}} = A\mathbf{x}$ in case A is the following matrix:

   (a) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, (b) $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, (c) $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

   Use Octave to draw phase portraits (solution trajectories for several initial values) for each of the systems above. How do eigenvalues of the matrix A affect the behaviour of the solutions?

2. The basic SIR compartmental model for modelling epidemics divides the population of size N into 3 compartments: susceptibles S, infectious I, and recovered R. The dynamics of this model is governed by the system of differential equations

$$\begin{aligned} \dot{S} &= -\frac{\beta I S}{N}, \\ \dot{I} &= \frac{\beta I S}{N} - \gamma I, \\ \dot{R} &= \gamma I, \end{aligned}$$

   where $S(t)$, $I(t)$, and $R(t)$ are functions of time, while $\dot{S} = \frac{dS}{dt}$, $\dot{I} = \frac{dI}{dt}$, and $\dot{R} = \frac{dR}{dt}$ are their derivatives. (The quotient $\frac{\beta}{\gamma}$ is often denoted by R_0 – basic reproduction number.)

   (a) Confirm that $\dot{S} + \dot{I} + \dot{R} = 0$, hence $S(t) + I(t) + R(t) = N$ (a constant – the size of population N).

   In what follows we assume $N = 1$, the functions S, I, and R therefore represent respective *fractions* of the population.

   (b) Show that $R(t) = 1 - S(t) - I(t)$, hence (w.r.t. solving the problem) only the first two equations of the above system are needed.

   (c) Use `rk4` and find the solutions of the system above for some chosen initial conditions and some chosen values of parameters β and γ.

3. *A mathematical pendulum* is a point mass m suspended on a massless (and inflexible) rod of length ℓ attached to a frictionless pivot. The mass m is acted upon by gravity mg, the displacement angle at time t is denoted by $\phi(t)$.

   (a) Show that ϕ solves the differential equation

$$\ddot{\phi} + \frac{g}{\ell}\sin(\phi) = 0.$$

   (b) Substitute $\omega = \dot{\phi}$ to convert the equation of order 2 above into a system of two order 1 equations.

   (c) Plot the phase diagram $(\phi, \dot{\phi}) = (\phi, \omega)$ of the equation above. Use `rk4`.