Nonlinear least squares method

Assume that we have a nonlinear function $\mathbf{F}\colon U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ where $m > n$, and we would like to solve the system

$$\mathbf{F}(\mathbf{x}) = \mathbf{0}.$$

Since the number of equations is greater than the number of unknowns such a system in general admits no solution. As in the case of linear systems we ask ourselves: What would be the best approximation to a solution? A good choice (as in the case of linear systems) is to find some $\mathbf{x} \in U$ which minimises the sum of squares $\|\mathbf{F}(\mathbf{x})\|^2$. An efficient way to find a solution of a nonlinear least–squares problem is the *Gauss-Newton* iteration, of which one step can be written as

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \left(J\mathbf{F}(\mathbf{x}^{(k)})\right)^+ \mathbf{F}(\mathbf{x}^{(k)}).$$

(Unlike Newton's method, where we find the solution $\mathbf{z}$ to the linear system $(J\mathbf{F}(\mathbf{x}^{(k)}))\mathbf{z} = \mathbf{F}(\mathbf{x}^{(k)})$, we find the solution to the *linear least–squares problem*. We have written the term $(J\mathbf{F})^+$ above just as a compact way of remembering the actual iteration step. The method only works well for overdetermined systems with full rank for which there is no need to actually evaluate the Moore–Penrose pseudo-inverse.) More precisely, the Gauss–Newton iteration will (with a little luck) produce a local minimum of the function $\mathbf{x} \mapsto \|\mathbf{F}(\mathbf{x})\|^2$.

1. The data

x_i	0.038	0.194	0.425	0.626	1.253	2.500	3.740
y_i	0.050	0.127	0.094	0.2122	0.2729	0.2665	0.3317

is to be modelled with a function of the form

$$f(x) = \frac{ax}{b+x}.$$

We will find the parameters a and b by the least squares method.

(a) Find the solution using Newton's method by differentiating the function $\mathbf{x} \mapsto \|\mathbf{F}(\mathbf{x})\|^2$ and finding the 'zero of the derivative'.

(b) Find the solution using the Gauss–Newton method.

2. We will solve the exercise from the first week's problem sessions: We have n known transmitter positions $(p_1, q_1), \ldots, (p_n, q_n)$ in $\mathbb{R}^2$. We want to locate a receiver which can measure its distances $d_1, \ldots, d_n$ form the transmitters using the signals from the transmitters. In the ideal case we would have exact measurements and we simply need to find the solution to the system of equations

$$(x - p_i)^2 + (y - q_i)^2 = d_i^2.$$

where $i = 1, \ldots, n$.

Improve the function `X = sprejemnik([pi, qi], [di])` from the first week using the Gauss–Newton method. The solution to the linear least squares problem we found can serve as the initial guess $\mathbf{x}^{(0)}$.